Dynamic policy for idling time preservation

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Abstract

This study aims to determine and evaluate dynamic idling policies where an agent can idle while some customers remain waiting. This type of policies can be employed in situations where the flow of urgent customers does not allow the agent to spend sufficient time on back-office tasks. We model the system as a single-agent exponential queue with abandonment. The objective is to minimize the system's congestion while ensuring a certain proportion of idling time for the agent. Using a Markov decision process approach, we prove that the optimal policy is a threshold policy according to which the agent should idle above (below) a certain threshold on the queue length if the congestion-related performance measure is concave (convex) with respect to the number of customers present. We subsequently obtain the stationary probabilities, performance measures, and idling time duration, expressed using complex integrals. We show how these integrals can be numerically computed and provide simpler expressions for fast-agent and heavy-traffic asymptotic cases. In practice, the most common way to regulate congestion is to control access to the service by rejecting some customers upon arrival. Our analysis reveals that idling policies allow high levels of idling probability that such rejection policies cannot reach. Furthermore, the greatest benefit of implementing an optimal idling policy occurs when the objective occupation rate is close to 50% in highly congested situations.

Keywords: queueing; idling time; Markov decision process; abandonment; optimization

1 Introduction

Service systems, such as call centers or hospitals, commonly encounter situations of high demand, leading to high congestion and high agent utilization. These situations may arise due to an under staffing decision, a peak of demand, or simply the variability of arrival and service processes. High agent utilization is usually viewed as an efficient use of resources since agents rarely idle during their work shifts. However, too high a rate of agent utilization may also result in fatigue, burn-out, low quality of service interaction, and inappropriate decisions (Castanheira and Chambel, 2010). For instance, Allan et al. (2019) analyzed a contact center where nurses were in charge of the routing of patients. They demonstrated that the longer nurses work without rest breaks, the more frequently they make the decision to arrange for callers to see another health professional the same day, which consequently leads to a deterioration of the system's performance. On the contrary, actively idling or taking breaks may reduce tiredness and allow employees to bond with their colleagues, which, in turn, may be beneficial for the company. For example, Waber et al. (2010) confirmed the hypothesis that the strength of an individual's social group was positively related to productivity measured by the average call handle time. Furthermore, agents in charge of urgent customers arriving over time may also be asked to spend time on back-office tasks (e.g., emails). Due to the preemptive priority of urgent customers over back-office tasks, the time spent on the latter may be insufficient to maintain the efficient functioning of the service.

Therefore, even in congested situations, it may be necessary for an agent to interrupt the flow of services to preserve a certain proportion of idling time. In our context, idling refers to time not spent on serving urgent customers. It can encompass time spent on back-office tasks. The academic literature has previously focused on the scheduling of breaks using static decisions based on past information with time-dependent parameters (Bechtold et al., 1984; Janaro and Bechtold, 1985; Alfares, 2007; Lujak and Billhardt, 2017). We refer the reader to the recent survey of Xu and Hall (2021) for an overview of fatigue management with a focus on work-rest scheduling. Other studies have considered the dynamic scheduling of idling time for the purpose of optimizing certain operational performance measures although without including idling time in the objectives (see Section 2).

The aim of this study is to characterize and compute the optimal dynamic idling policy for an agent such that certain congestion-related performance measures are minimized while maintaining the proportion of busy time below a threshold level. We analyze this problem for a single-agent exponential queue with impatient customers. At each service completion, the agent may decide either to continue serving customers or to remain idle. When the agent is idling, service initiation can start at a chosen moment. The formulation of the congestion cost function is general. Specifically, it can be adjusted to moments of the number of customers in the queue, to the rate and moments of abandonment, or to the expected excess (i.e., the expected wait above a given threshold).

To the best of our knowledge, the optimal policy for this problem has not been investigated in the academic literature. One reason is that due to flow conservation, the proportion of busy time and the expected number of customers in the queue are related in a linear way. Specifically, in an infinite capacity queue, the arrival rate is equal to the rate of abandonment plus the rate of served customers. The rate of abandonment is proportional to the expected number of customers in the queue. Also, the rate of served customers is proportional to the proportion of busy time of the agent. Therefore, for an optimization problem involving a linear performance measure, any policy that assigns a certain value to the proportion of busy time would achieve the same operational performance measure. Consequently, the policy optimization question is irrelevant with linear performance measures.

However, for nonlinear performance measures that are used in practice, such as waiting time percentiles (Legros, 2016), expected excess (Koole, 2013), or measures that involve moments of the queue length such as fairness measures (Avi-Itzhak et al., 2008) or queue length distribution (Shore, 2006), the dynamic idling policy matters. It is thus essential to determine the best way to achieve a certain level of idling time while limiting the impact on the operational performance measures. To address this issue, we formulate the optimization problem as a Markov decision process. We prove structural properties of the value function that lead to the form of the optimal policy. When the service rate is greater than or equal to the abandonment rate, we prove that the optimal idling policy is of threshold type when the cost function is either convex or concave with respect to the number of customers in the queue. When the cost function is convex (concave), there exists a threshold on the queue length such that idling is optimal below (above) this threshold. The concave case contradicts the intuition according to which the decision to work is incentivized by the number of waiting customers. We also prove that idling above a threshold is optimal for some nonconcave cost functions when the abandonment rate is greater than or equal to the service rate.

Next, we evaluate the performance measures under the two aforementioned policies. In the evaluation, we also consider a nonoptimal threshold-type reference policy. For this reference policy, the agent works until the system is empty. Once it is empty, the agent waits until the queue size reaches a given threshold level to start serving customers. Although nonoptimal, this policy has the advantage of not letting the agent leave any customers waiting in the queue since this would not be appreciated in visible systems. It also reduces the frequency of switches between idle and busy states for the agent (Legros et al., 2020). The stationary probabilities are computed via a two-dimensional Markov chain analysis. The formulas are obtained in closed form. However, some complex integrals are involved in the evaluation. Using the asymptotic values of these integrals and their Wronskian formulation, we determine a numerical way to compute them. Next, we determine the idling duration. The evaluation involves confluent hypergeometric functions of the first and second kind (Daalhuis, 2010).

Our numerical experiments show that in most cases, decreasing the agent's occupation rate increases the system's congestion due to the lack of work conservation when the agent idles while the queue is not empty. In addition, we illustrate that idling above a certain threshold is optimal for concave performance measures, while the opposite is true for convex ones. In both cases, the optimal policy provides significant improvements as compared to the reference policy when the agent expects to maintain around 50% idling time in a highly congested situation. When the idling-time proportion is far from 50%, either due to the agent's decision or to the system's congestion, the dynamic idling policy does not significantly affect the operational performance, leading to a reduced difference between the different idling policies. We further observe that the operational performance is convex with respect to the agent's occupation rate under the optimal policy. This means that if the dynamic idling policy is well selected, increasing the proportion of idling time may only have a small detrimental impact on operational performance. We also illustrate that the main advantage of the reference policy is to allow the agent to have longer breaks compared to the break duration under the optimal policy. Next, we provide approximations for the probability of the agent being busy. We consider fast-agent, high-congestion and Normal approximations. These approximations are useful as they lead to simpler expressions for the idling probability and simpler computation of the optimal threshold level. Finally, we compare the idling policies considered in this study with a rejection policy, where some customers are rejected upon arrival with the aim of reducing the system's congestion. Rejection policies are the most commonly implemented and studied policies to obtain a good trade-off between the system's congestion and the rate of served customers (Lin and Ross, 2004; Koçağa and Ward, 2010; Su et al., 2019). The comparison reveals that idling policies can only be implemented when the objective occupation rate is low, whereas a rejection policy achieves better performance measures when the desired idling probability is low.

Structure of the paper. Section 2 provides a review of the related academic literature. Section 3 presents the formulation of the model and optimization question. Section 4 determines the optimal policy from a Markov decision process approach. In Section 5 derives the stationary probabilities and performance measures. Section 6 provides numerical experiments and approximations. Finally, with Section 7 the paper is concluded and avenues for future research provided. A table of notation and the proofs of the main results are given in the appendix at the end of the paper.

2 Literature review

First, since our study considers the optimization of idling decisions, we examine prior studies in this field that have addressed different optimization questions than ours. Next, in relation to the possible multitasking applications of our analysis, we present the existing literature on blended queues. Finally, we present an overview of queueing models involving breaks and vacations.

Different studies have revealed that it may be optimal to maintain some agents idling while the queue is not empty. In other words, work conservation is shown to be not always optimal, as in the current paper. For instance, in the slow-server problem, it may be optimal to let the slowest agents idle, while nonidling is optimal for the fastest agent for the purpose of minimizing the expected time spent in the system (Koole, 1995; Cabral, 2005; Rykov and Efrosinin, 2009; Özkan and Kharoufeh, 2014). Also, when customers are heterogeneous, it may be optimal to implement strategic delays (e.g., for the most patient customers) with the aim of maximizing the system's revenue (Afèche, 2013; Afèche and Pavlin, 2016; Maglaras et al., 2018). For service rate optimization problems where a trade-off between a holding cost and mean service rate has to be determined, previous studies showed that the optimal state-dependent service rate is located at the boundary of the value domain (i.e., a bang-bang control) (Ma and Ao, 1994; Kumar et al., 2013; Xia, 2014; Xia et al., 2017). In particular, if zero is at the boundary of the service rate's interval, idling is optimal in some states as we prove in this study. Finally, with a single class of homogeneous customers, Zhan and Ward (2019) and Zhong et al. (2022) have shown that intentional idling may arise as an asymptotically optimal regime where customers may wait in a holding area before joining the queue when a trade-off between agent utilization costs and operational costs has to be determined. These references illustrate the value of non-work-conserving policies for solving various optimization problems. However, this study differs from the aforementioned references as we characterize and prove the form of the optimal policy in the general regime for an optimization problem involving a single class of impatient customers.

The main idea in queue-blending models is to determine efficient scheduling policies for the treatment of urgent and nonurgent jobs. The optimization problem in relevant studies consists of maximizing the time spent on nonurgent jobs while imposing a service level constraint on urgent ones. In this context, the idling decision is referred to as a *reservation* strategy (Bhulai and Koole, 2003; Pang and Perry, 2014; Legros, 2017; Legros et al., 2021; Legros, 2021). This implies that the flow of urgent customers allows for sufficient idling times in order to treat nonurgent tasks. This is an important difference with our context, where the flow of urgent customers does not allow the agent to spend a sufficient amount of time on back-office tasks. Blending models have been widely studied in the context of call centers. Models by Brandt and Brandt (1999) and Deslauriers et al. (2007) evaluated performance in such systems. Bhulai and Koole (2003) and Gans and Zhou (2003) considered queue-blending models in which the inbound jobs have a nonpreemptive priority over the outbound ones. They showed that the optimal policy is a reservation threshold policy on the number of busy agents. This also makes a difference with our context, where urgent customers can interrupt the treatment of some back-office tasks. In a context with switching time allocation between urgent and nonurgent tasks, Legros et al. (2020) showed that idling in the presence of waiting customers may be optimal to avoid too many switches. Further references on queue-blending operations include Keblis and Chen (2006); Pichitlamken et al. (2003); Pang and Perry (2014).

Some academic research has focused on the performance analysis of queues with vacations. Fuhrmann (1984) decomposed the number of customers in an M/G/1 queue in which the agent begins a vacation of random length each time the system becomes empty. This model is close to the reference policy considered in this study. The difference is that the vacation time is controlled in our study. Later, Kella (1989) dealt with the M/G/1 queue with agent vacations in which the return of the agent to service depends on the number of customers present in the system, as in the reference policy in this study. The authors determined expressions that characterize the optimal number of customers, below which the agent should not start a new service period. Li et al. (1997) considered an M/G/1 queue with Bernoulli vacations and agent breakdowns. Using a supplementary variable method, they obtained a transient solution for queueing and reliability measures of interest. Chao and Zhao (1998) investigated a GI/M/c queue with two classes of vacation mechanisms: station vacation and agent vacation. For both models, they derived steady-state probabilities of matrix geometric form, and developed computational algorithms to obtain numerical solutions. Ke (2003) studied a queue with the decision-maker able to turn a single agent on at any arrival epoch or off at any service completion. When the system is empty, the agent takes a vacation of exponential random length. They derived the distribution of the system size using the probability-generating function. Altman and Yechiali (2006) analyzed a systems with agents vacations and customer impatience, as in this study. They presented a comprehensive analysis of the single-agent, M/M/1, and M/G/1 queues, as well as of the multi-agent M/M/c queue, for both the multiple- and single-vacation cases. Finally, Zhang et al. (2020) investigated a queueing-inventory system under continuous review with a random order size policy and lost sales, which can be modeled as a queue with agent vacations. They derived the stationary joint distribution of the queue length, the on-hand inventory level, and the status of the agent in explicit product form. To the best our knowledge, the optimal policies determined in this study have not yet been investigated as a way to operate breaks for the agent.

3 Formulation of the problem

In this section, we explain the model assumptions and formulate the optimization question. We consider a single-agent queue with infinite capacity where customers are served in the order of their arrival. The customers' arrival process is Poisson with rate λ, and service times are exponentially distributed with rate μ. Furthermore, customers in the queue (excluding the one in service) have limited patience. Their patience time is exponentially distributed with rate γ. As such, the considered queueing model is an M/M/1+M queue, in accordance with Kendall's notation.

For this queueing system, we consider a wide set of congestion related performance measures, which may evaluate the service quality offered to customers. This includes the expected waiting time, the rate of abandonment, or the moment generating function of the number of customers in the queue. To capture the diversity of service quality measures, we introduce a cost function $f_S(N)$ that depends on the stationary random variables N and S, corresponding to the number of customers waiting in the queue and the status –idle ($S = I$) or busy ($S = B$)– of the agent, respectively. The set of values for N and S are $\mathbb{N}_0$ and $\{I, B\}$, respectively. In Table 1, we provide a nonexhaustive list of possible expressions for the expected value of the cost function $f_S(N)$, termed $E(f_S(N))$. It should be noted that cost functions that depend on

Table 1: Expected cost function $E(f_S(N))$

$E(f_S(N))$	Performance measure
$E(N)$	Expected number of customers in the queue
$E(N^k)$	k^{th} moment of N
$E\left(\frac{N}{\lambda}\right) = E(W)$	Expected waiting time
$E(\gamma N)$	Expected rate of abandonment
$E(e^{tN})$	Moment generating function of N at t
$E((N - E(N))^+)$	Expected excess in the number of customers in the queue
$E(\mathbb{1}_{S=I} N)$	Expected number of customers in the queue when the agent is idling
$E(\mathbb{1}_{S=B} N)$	Expected number of customers in the queue when the agent is busy

the agent's status correspond to visible queues where customers may perceive the wait differently whether the agent is actively serving them or not. When the function $f_S(N)$ does not depend on the status of the agent, we write $f(N)$ instead of $f_S(N)$. We are also concerned with the work condition offered to the agent. Therefore, we evaluate the proportion of time during which the agent is busy (that is, the occupation rate), termed p_B.

The objective is to obtain a trade-off between the service quality offered to customers and work condition offered to the agent. To capture this trade-off, we wish to minimize the expected congestion cost function while having the proportion of time during which the agent is busy below a certain threshold level. The optimization problem is then expressed as

$$\begin{cases} \text{Minimize } E(f_S(N)) \\ p_B \leq \overline{p_B}, \end{cases} \tag{1}$$

where $\overline{p_B}$ is the threshold level for p_B, corresponding to the maximal occupation rate of the agent.

To solve Problem 1, we consider the set of non-work-conserving policies. This means that we consider policies that allow the agent to remain idle even if a customer is waiting in the queue. Therefore, the agent controls the proportion of busy time by actively not working when some customers are waiting to be served. However, we do not allow the agent to interrupt a service and put a customer back into the queue to become idle once a service has started. Allowing this possibility could enlarge the set of achievable policies, eventually leading to a better solution for Problem 1, but it may not correspond to service system applications where service interruptions are not permitted.

4 Characterization of the optimal policy

In this section, we develop a Markov decision process (MDP) approach to determine the optimal idling policy for Problem 1. The optimization problem corresponds to a constrained MDP. Constrained MDPs can be solved using various techniques such as linear programming (Altman, 1998), reinforcement learning (Geibel, 2006) or genetic algorithms (Hirayama and Kawai, 2000). The selection of the appropriate method depends on the dimensionality of the underlying Markov chain and the aim to either compute or prove the form of the optimal policy. For this problem, we use the Lagrangian method explained in Altman (1999) that introduces a Lagrangian multiplier $P > 0$ to measure the proportion of time during which the agent is busy. Therefore, the MDP approach will determine the optimal policy to minimize $P \times p_B + E(f_S(N))$. This optimization problem is not equivalent to Problem 1. However, the optimal policy for this new optimization problem has the same monotonic structure as the one for the constrained Problem 1 (e.g., see Chapter 5 in Altman (1999)). Once the monotonic structure of the optimal policy is determined, a performance evaluation method will be developed in Section 5 to obtain the solution to Problem 1. We now formulate the problem via the definition of states, transition structure and possible actions.

State definition. We model the system in a Markov theoretic framework. Let (x, y) be a state of the Markov chain associated with the number of customers in the queue (excluding the one in service) and the status of the agent. As defined in Section 3, the status of the agent is denoted by either I or B to indicate that the agent is either idle or busy. Therefore, the state space of the system is $\mathbb{N}_0 \times \{I, B\}$.

Transition structure. The transition structure is that of an M/M/1+M queue, except that there is no automatic routing of one customer in service. This is a decision action as explained below. We denote the transition rate from state (x, y) to state (x', y') by $r_{(x,y),(x',y')}$. Then for (x, y), $(x', y') \neq (x, y) \in \mathbb{N}_0 \times \{I, B\}$, we have

$$r_{(x,y),(x',y')} = \begin{cases} \lambda & \text{if } (x',y') = (x+1,y), \\ \mu & \text{if } y = B \text{ and } (x',y') = (x,I), \\ x\gamma & \text{if } x > 0 \text{ and } (x',y') = (x-1,y), \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Control action. At each instant of time, when (i) at least one customer is in the queue (i.e., $x > 0$) and (ii) the agent is idling (i.e., $y = I$), the agent can take the decision action to remain idle or to initiate a new service. An *idling policy* is a function that associates either state (x, I) or state $(x - 1, B)$ to each state (x, I) with $x > 0$.

Value function definition. We are considering infinite horizon average costs. It is then optimal to schedule customers only at transition instants (abandonment instant, service completion time or arrival time). Specifically, if it is optimal to keep the agent idle at a given time, then the action remains optimal until the next event in the system. This result follows directly from the continuous-time Bellman equation (Puterman (1994), Chapter 11). We then choose to discretize our continuous-time model. However, since we consider an infinite capacity queue, the total event rate is unbounded due to the abandonment rate. Therefore, we cannot apply the uniformization technique directly. We thus introduce a bound for the number of customers in the queue, m, such that the total event rate is bounded by $\lambda + \mu + m\gamma$. We further assume that $\lambda + \mu + m\gamma = 1$ such that the transition rates in the continuous time MDP can be viewed as transition probabilities in the discrete time MDP. In Sections C4 and C5 of Sennott (2009), it is proven that the average expected cost and optimal policy under the truncated model converges to the average expected cost and optimal policy of the original model as m tends to infinity. Therefore, by selecting a sufficiently high value for m, the truncated finite state MDP approximates the real system.

We formulate a two-step value function, in order to separate transitions and actions and simplify the involved expressions. We define the dynamic programming value functions $V_k(x, y)$ and $W_k(x)$ over $k \geq 0$ steps, depending on the state of the system. The operator W_k captures the decision action to either remain idle or to start the service of a new customer. Therefore,

$$W_k(x) = \min(V_k(x-1,B), V_k(x,I)) \text{ if } x > 0, \text{ and} \tag{3}$$
$$W_k(0) = V_k(0,I).$$

We mention that the operator W_k cannot be used after a $\lambda-$ or a $\gamma-$transition from state $y = B$ as service interruption is not permitted.

We next express $V_{k+1}(x,y)$ in terms of $V_k(x,y)$ in the following way. First, the costs until the next jump are incurred by the congestion cost function $f_y(x)$ defined as in Section 3. Second, a cost P is counted for each state where the agent is busy, to account for the occupation rate. For $0 \leq x < m$ and $k \geq 0$, this leads to

$$\begin{aligned} V_{k+1}(x,I) &= f_I(x) + \lambda W_k(x+1) + \gamma x W_k(x-1) + (1-\lambda-\gamma x) W_k(x), \text{ and} \\ V_{k+1}(x,B) &= f_B(x) + P + \lambda V_k(x+1,B) + \mu W_k(x) + \gamma x V_k(x-1,B) + (1-\lambda-\mu-\gamma x) V_k(x,B). \end{aligned} \tag{4}$$

Note that by adding a fictitious transition from a state to itself (i.e., the rates $1-\lambda-\gamma x$ and $1-\lambda-\mu-\gamma x$), we allow the rate out of each state to be $\lambda+\mu+m\gamma = 1$, without exception, for each state.

There remains to express $V_{k+1}(x,y)$ in terms of $V_k(x,y)$ at the boundary state $x = m$. As mentioned above, when m is selected sufficiently high, the transitions at the boundary state do not influence the optimal policy and average cost. Therefore, we could simply express in the definition of $V_k(m,y)$ that, at the boundary state $x = m$, a $\lambda-$transition does not modify the system state. However, this would break the monotonicty properties of $V_k(x,y)$ in x of the original model at the boundary state. Instead of neglecting this aspect (assuming that m is very large), we propose to modify the transitions at the boundary state in order to keep the monotonicity structure that is hidden in the original model, intact as in Down et al. (2011); Bhulai et al. (2014). We propose the following relation for $x = m$ and $k \geq 0$:

$$\begin{aligned} V_{k+1}(m,I) &= f_I(m) + \lambda V_k(m,B) + \gamma m W_k(m-1) + (1-\lambda-\gamma m) W_k(m), \text{ and} \\ V_{k+1}(m,B) &= f_B(m) + P + \lambda(2V_k(m,B) - W_k(m-1)) + \mu W_k(m) + \gamma m V_k(m-1,B) \\ &\quad + (1-\lambda-\mu-\gamma m) V_k(m,B) \end{aligned} \tag{5}$$

Theorem 6.2.3 in Puterman (1994) proves that the optimal infinite horizon policy is independent of the choice of $V_0(x,y)$. We thus simply select $V_0(x,y) = 0$ for $0 \leq x \leq m$ and $y = I, B$ to initiate the definition of the value function.

For each $k > 0$ and each state (x,I) for $0 < x \leq m$, there is a minimizing action: serve a customer or remain idle. The function

$$(x,I) :\rightarrow \{\text{Serve}, \text{Idle}\}$$

is referred to as the agent's idling policy at iteration k. As k tends to infinity, the difference $V_{k+1}(x,y) -$

$V_k(x, y)$ converges to the optimal long-run average cost, and the optimal action at each state can be found (Puterman, 1994). Therefore, (4) and (5) can be used to obtain the optimal policy numerically. In addition, these relations allow us to prove the monotonicity properties of the value function $V_k(x, y)$ that show the form of the optimal policy.

Main result. The main result of this section is that it is either optimal to idle above or below a threshold on the queue length. In Lemma 1, we first state necessary relations for $V_k(x, y)$ that induce the result for the optimal idling policy at iteration k. We aim to prove by induction on k that (6) in Lemma 1 holds for $V_k(x, y)$ using (4) and (5). In the induction step, we observe that (6) does not simply propagate. Other relations are required to prove the induction step. These relations define three classes of functions that include those of Lemma 1. Let $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ denote these different sets of relations for a function $g_y(x)$ for $0 \leq x \leq m$ and $y = I, B$. In Table 2, we define these sets by using the symbol ✓ when one property is satisfied in one class of functions. In Theorem 1, under some conditions on the system parameters, we prove by induction on k that if the congestion cost function $f_y(x)$ belongs to $\mathcal{C}_i$, then the properties defining the set $\mathcal{C}_i$ propagates in the induction step from $V_k(x, y)$ to $V_{k+1}(x, y)$ using (4) and (5) for $i = 1, 2, 3$. In particular, this proves that (6) propagates, which in turn proves the form of the optimal idling policy.

Lemma 1. *At iteration $k \geq 0$, if for $0 < x < m$ we have*

$$V_k(x+1, I) + V_k(x-1, B) - V_k(x, B) - V_k(x, I) \geq 0 \ (\leq 0), \tag{6}$$

then it is optimal for the agent to idle below (above) a threshold on the number of customers in the queue.

Table 2: Definition of the classes of function $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ for $0 \leq x \leq m$ and $y = I, B$

Property	Definition	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
Increasing in x	$g_y(x+1) - g_y(x) \geq 0$	✓	✓	✓
Increasing in y	$g_B(x) - g_I(x) \geq 0$	✓	✓	✓
Convexity in x	$g_y(x+1) + g_y(x-1) - 2g_y(x) \geq 0$	✓	✓	
Supermodularity in (x, y)	$g_B(x) + g_I(x-1) - g_B(x-1) - g_I(x) \geq 0$	✓	✓	
Concavity in x	$g_y(x+1) + g_y(x-1) - 2g_y(x) \leq 0$			✓
Submodularity in (x, y)	$g_B(x) + g_I(x-1) - g_B(x-1) - g_I(x) \leq 0$			✓
Idle below a threshold	$g_I(x+1) + g_B(x-1) - g_B(x) - g_I(x) \geq 0$	✓		
Idle above a threshold	$g_I(x+1) + g_B(x-1) - g_B(x) - g_I(x) \leq 0$		✓	✓
Complementary property	$g_I(x+1) + g_B(x-2) - g_I(x) - g_B(x-1) \geq 0$		✓	

Theorem 1. *The optimal policy for Problem 1 is determined in the following cases:*

- *Policy π_1: If $\mu \geq \gamma$ and $f_y(x) \in \mathcal{C}_1$, there exists a threshold n on the number of customers in the queue such that, idling is optimal if and only if $x \leq n$.*

- *Policy π_2: If $\gamma \geq \mu$, $\lambda \geq 2\gamma$ and $f_y(x) \in \mathcal{C}_2$ or $\mu \geq \gamma$ and $f_y(x) \in \mathcal{C}_3$, then there exists a threshold n on the number of customers in the queue such that, idling is optimal if and only if $x \geq n$.*

Remarks. We end this section with some remarks regarding the proof and the form of the optimal policies.

- **Properties of the cost function.** To prove Theorem 1, we prove that if the congestion cost function $f_y(x)$ belongs to $\mathcal{C}_i$, then the value function $V_k(x, y)$ also belongs to $\mathcal{C}_i$ for $i = 1, 2, 3$ for $k \geq 0$. We prove the result by showing the propagation in k of the different relations defining the classes $\mathcal{C}_i$ for $i = 1, 2, 3$. The two first properties in Table 2 indicate that the functions in $\mathcal{C}_i$ are increasing in x and in y for $i = 1, 2, 3$. This translates the idea that the cost of the system increases with the number of customers present. The next fourth rows give the second order monotonicity properties (convexity, concavity, supermodularity and submodularity). Rows 7 and 8 provide the relations that define the threshold structure of the optimal policy, that is those of Lemma 1. Finally, the last line defines a complementary property that is required for $\mathcal{C}_2$. In the case where $g_I(x) = g_B(x)$ (i.e., if $g_y(x)$ does not depend on y), then $\mathcal{C}_1$ is the set of increasing and convex functions in x, $\mathcal{C}_2$ is the set of linear and increasing functions in x, and $\mathcal{C}_3$ is the set of increasing and concave functions in x. These properties can be related to those presented in Table 1. For instance, $\mathcal{C}_1$ ($\mathcal{C}_3$) translates performance measures such as the k^{th} moment of the queue length with $k \geq 1$ ($0 \leq k \leq 1$), while $\mathcal{C}_2$ captures linear performance measures such as the expected wait or rate of abandonment.
  It is difficult to provide an intuition that could explain in general the form of the optimal policy from the definition of the congestion cost function. When the function $f_y(x)$ does not depend on y, it is however possible to understand whether it would be advisable to idle below or above a certain threshold on the queue length. When $f_y(x)$ is convex in x, the marginal cost per customer is increasing in the number of customers in the queue. Therefore, long queues have costlier customers than short ones. It is then advisable to idle only when the queue size is short. The opposite is true when $f_y(x)$ is concave in x. Short queues have costlier customers than long ones. Therefore, it is advisable to idle only when the queue size is long.

- **Conditions on the system parameters.** To show the induction step, we need to add conditions on the system parameters such as $\mu \geq \gamma$ for $\mathcal{C}_1$ and $\mathcal{C}_3$, and $\gamma \geq \mu$ and $\lambda \geq 2\gamma$ for $\mathcal{C}_2$. These conditions arise from the induction step in the proof of Theorem 1 and are necessary to show the propagation in k of the considered properties. When these conditions are not satisfied, we observe that for certain values of k, $V_k(x, y)$ does not belong to any of the $\mathcal{C}_i$'s. However, as k tends to infinity, we observe that the desired monotonicity properties hold without conditions on the system parameters. It should be noted that such conditions on the system parameters are often needed to prove monotonicity results for queueing systems. For instance, the condition $\mu \geq \gamma$ in the first and third statement of Theorem 1 was also required in Armony et al. (2009) to prove the convexity of the expected queue length in μ for an M/M/s+M queue.

5 Performance evaluation

In this section, we determine the stationary performance measures for Policy π_1 and Policy π_2. We also consider a reference policy, termed Policy π_0. Policy π_0 is controlled by a threshold n on the queue size such that an idle agent starts serving customers only once an arrival occurs if the queue size has reached n, and a busy agent serves customers until the system becomes empty. In Section 5.1, we determine explicit expressions of the stationary probabilities that allow computing the performance measures. Next in Section 5.2, we explain how one of the building blocks that is part of the stationary probabilities can be evaluated numerically. Finally in Section 5.3, we focus on the expected idling time duration.

5.1 Stationary probabilities

In this section, we determine the stationary probabilities for each policy. To explain the dynamics of each policy, we first describe their transition structure. A state of the system is defined as in the previous section by the pair (x, y) where x is the number of customers in the queue and y is the status of the agent. We denote the transition rate from state (x, y) to state (x', y') by $r^i_{(x,y),(x',y')}$ for Policy π_i with $i = 0, 1, 2$. For Policy π_0 with (x, y), $(x', y') \neq (x, y) \in \mathbb{N}_0 \times \{I, B\}$, we have

$$r^0_{(x,y),(x',y')} = \begin{cases} \lambda & \text{if } x \geq 0 \text{ and } y = B \text{ or if } 0 \leq x < n \text{ and } y = I, \quad \text{with } (x', y') = (x + 1, y), \\ \lambda & \text{if } x = n \text{ and } y = I, \quad \text{with } (x', y') = (n, B), \\ \mu & \text{if } x > 0 \text{ and } y = B, \quad \text{with } (x', y') = (x - 1, B), \\ \mu & \text{if } x = 0 \text{ and } y = B, \quad \text{with } (x', y') = (0, I), \\ x\gamma & \text{if } x > 0, \quad \text{with } (x', y') = (x - 1, y), \\ 0 & \text{otherwise.} \end{cases}$$

Policy π_1 differs from Policy π_0 in the third and fourth transition. Therefore, for (x,y), $(x',y') \neq (x,y) \in \mathbb{N}_0 \times \{I,B\}$, we deduce that

$$r^1_{(x,y),(x',y')} = \begin{cases} \lambda & \text{if } x \geq 0 \text{ and } y = B \text{ or if } 0 \leq x < n \text{ and } y = I, & \text{with } (x',y') = (x+1,y), \\ \lambda & \text{if } x = n \text{ and } y = I, & \text{with } (x',y') = (n,B), \\ \mu & \text{if } x > n \text{ and } y = B, & \text{with } (x',y') = (x-1,B), \\ \mu & \text{if } 0 \leq x \leq n \text{ and } y = B, & \text{with } (x',y') = (x,I), \\ x\gamma & \text{if } x > 0, & \text{with } (x',y') = (x-1,y), \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for Policy π_2 with (x,y), $(x',y') \neq (x,y) \in \mathbb{N}_0 \times \{I,B\}$, we get

$$r^2_{(x,y),(x',y')} = \begin{cases} \lambda & \text{if } x > 0 \text{ and } y = B \text{ or if } 0 \leq x < n \text{ and } y = I, & \text{with } (x',y') = (x+1,y), \\ \lambda & \text{if } x = 0 \text{ and } y = I, & \text{with } (x',y') = (0,B), \\ \mu & \text{if } x = 0 \text{ and } y = B, & \text{with } (x',y') = (0,I), \\ \mu & \text{if } 0 < x < n \text{ and } y = B, & \text{with } (x',y') = (x-1,B), \\ \mu & \text{if } x \geq n \text{ and } y = B, & \text{with } (x',y') = (x,I), \\ x\gamma & \text{if } x > 0 \text{ and } y = B \text{ or if } x > n \text{ and } y = I, & \text{with } (x',y') = (x-1,y), \\ n\gamma & \text{if } x = n \text{ and } y = I, & \text{with } (x',y') = (n-2,B), \\ 0 & \text{otherwise.} \end{cases}$$

Figure 1 presents the Markov chain associated with each policy.

For Policy π_i, we denote by p^i_x and q^i_x the stationary probabilities of having x customers in the queue and the agent being idle or busy, respectively, for $x \geq 0$ and $i = 0,1,2$. We introduce the notations $a = \frac{\lambda}{\gamma}$ and $s = \frac{\mu}{\gamma}$. We next provide the balance equations for each policy. For Policy π_0, we have

$$(a+s+x)q^0_x = aq^0_{x-1} + (x+1+s)q^0_{x+1} \text{ for } 0 \leq x \leq n-1, \tag{7}$$

$$ap^0_0 = sq^0_0 + p^0_1, \tag{8}$$

$$(a+x)p^0_x = ap^0_{x-1} + (x+1)p^0_{x+1} \text{ for } 1 \leq x \leq n, \tag{9}$$

$$(a+s+n)q^0_n = aq^0_{n-1} + (n+1+s)q^0_{n+1} + ap^0_n, \text{ and} \tag{10}$$

$$(a+s+x)q^0_x = aq^0_{x-1} + (x+1+s)q^0_{x+1} \text{ for } x \geq n+1, \tag{11}$$

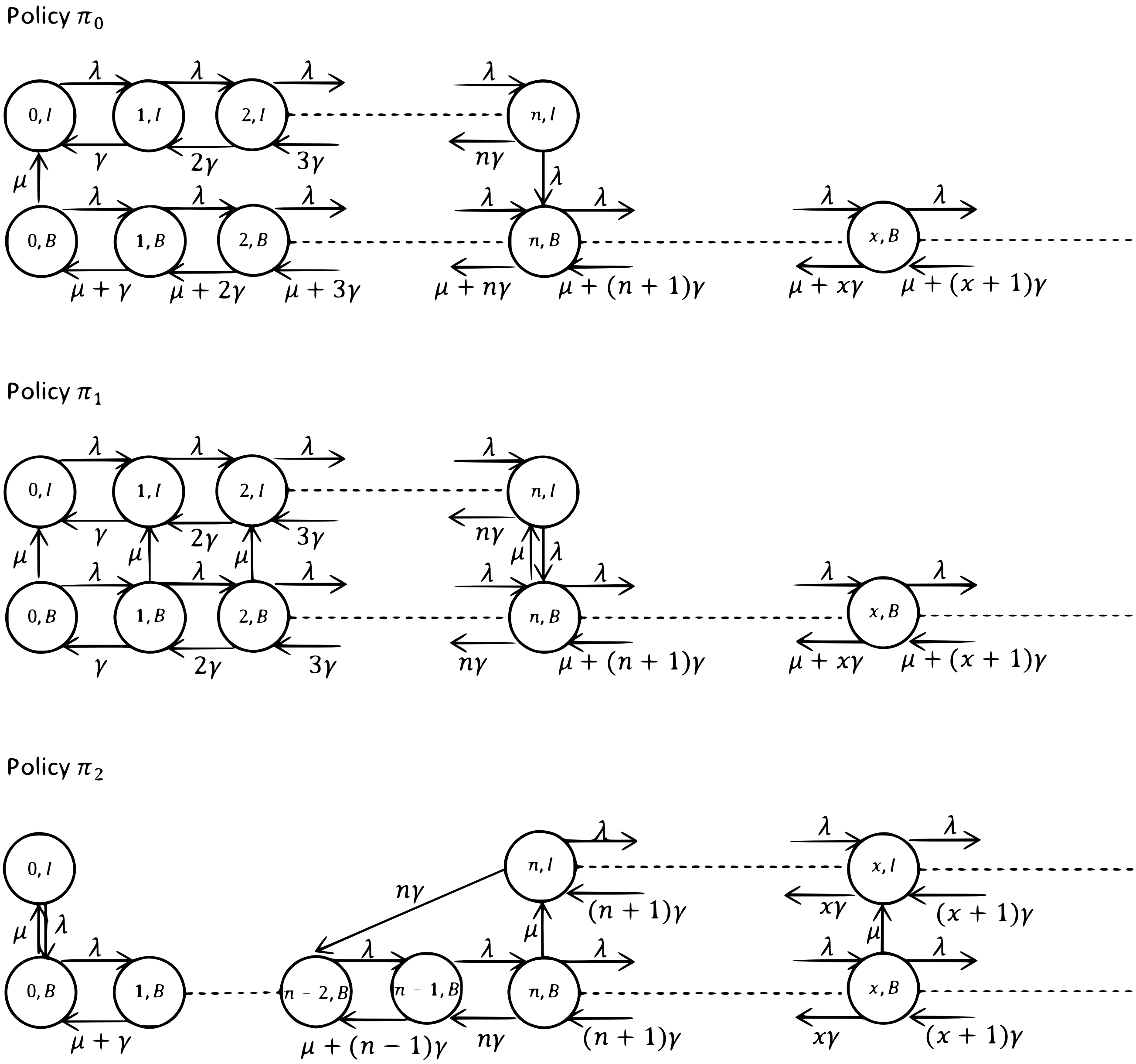


Figure 1: Markov chains for Policies π_0, π_1, and π_2.

with the convention $q^0_{-1} = p^0_{n+1} = 0$. For Policy π_1, we instead have

$$(a + s + x)q^1_x = aq^1_{x-1} + (x + 1)q^1_{x+1} \text{ for } 0 \le x \le n - 1, \tag{12}$$

$$(a + x)p^1_x = ap^1_{x-1} + (x + 1)p^1_{x+1} + sq^1_x \text{ for } 0 \le x \le n, \tag{13}$$

$$(a + s + n)q^1_n = aq^1_{n-1} + (n + 1 + s)q^1_{n+1} + ap^1_n, \text{ and} \tag{14}$$

$$(a + s + x)q^1_x = aq^1_{x-1} + (x + 1 + s)q^1_{x+1} \text{ for } x \ge n + 1, \tag{15}$$

with the convention $p^1_{n+1} = q^1_{-1} = p^1_{-1} = 0$. Finally, for Policy π_2, we get

$$ap^2_0 = sq^2_0, \tag{16}$$

$$(a + s + x)q^2_x = aq^2_{x-1} + (x + 1 + s)q^2_{x+1} \text{ for } 0 \le x \le n - 3 \text{ and } x = n - 1, \tag{17}$$

$$(a + s + n - 2)q^2_{n-2} = aq^2_{n-3} + (n - 1 + s)q^2_{n-1} + np^2_n \text{ for } x = n - 2, \tag{18}$$

$$(a + s + x)q^2_x = aq^2_{x-1} + (x + 1)q^2_{x+1} \text{ for } x \ge n, \text{ and} \tag{19}$$

$$(a + x)p^2_x = ap^2_{x-1} + (x + 1)p^2_{x+1} + sq^2_x \text{ for } x \ge n, \tag{20}$$

with the convention $q^2_{-1} = p^2_0$ and $p^2_{n-1} = 0$. We observe that common structures of equations arise for the different policies. Specifically, we observe that the following set equations is involved for a given probability w_x for $x \ge 0$:

$$(a + s + x)w_x = aw_{x-1} + (x + 1)w_{x+1}, \tag{21}$$

$$(a + x)w_x = aw_{x-1} + (x + 1)w_{x+1}, \text{ and} \tag{22}$$

$$(a + s + x)w_x = aw_{x-1} + (x + 1 + s)w_{x+1}. \tag{23}$$

In Lemma 2, we provide the two independent solutions of each equation (21), (22), and (23). These solutions are next used as building blocks to express the stationary probabilities in Theorem 2.

Lemma 2. *The two independent solutions of (21) for $x \ge 0$ are A_x and B_x, defined as follows:*

$$A_x = \frac{1}{2i\pi} \int_{\zeta_1} z^{-(x+1)} e^{az} (1 - z)^{-s} dz = \sum_{k=0}^{x} \frac{a^{x-k}}{k!(x - k)!} \frac{\Gamma(s + k)}{\Gamma(s)}, \text{ and} \tag{24}$$

$$B_x = \frac{1}{2i\pi} \int_{\zeta_2} z^{-(x+1)} e^{az} (z - 1)^{-s} dz, \tag{25}$$

where the contour ζ_1 is defined as a small circle in the z-plane, on which $|z| < 1$, the contour ζ_2 goes from

$-\infty - i\epsilon$ to $-\infty + i\epsilon$ for $\epsilon > 0$, encircling $z = 1$ in the counterclockwise sense, and $\Gamma(z)$ is the Gamma function defined for $z > 0$ by $\Gamma(z) = \int_{t=0}^{\infty} t^{z-1} e^{-t}\, dt$.

The two independent solutions of (22) for $x \geq 0$ are

$$C_x = \frac{a^x}{x!}, \text{ and} \tag{26}$$

$$D_x = \sum_{k=0}^{x-1} \frac{a^{x-k} k!}{x!}. \tag{27}$$

Finally, the two independent solutions of (23) for $x \geq 0$ are given by

$$E_x = \frac{a^{x+s}}{\Gamma(x+1+s)}, \text{ and} \tag{28}$$

$$F_x = \sum_{k=0}^{x} \frac{a^k s \Gamma(s+x-k)}{\Gamma(s+x+1)}. \tag{29}$$

Although B_x can be expressed as a complex integral, it cannot be numerically computed directly. To solve this issue, Section 5.2 proposes two numerical methods to derive B_x. These methods are built on the Wronskian of A_x and B_x, defined by $U_x = A_x B_{x-1} - B_x A_{x-1}$ for $x \geq 0$. It should be noted that the definition of A_x and B_x is then extended to $x = -1$ by computing their values at $x = -1$. We find that $A_{-1} = 0$ and $B_{-1} = \frac{e^a a^{s-1}}{\Gamma(s)}$ from the expressions in Lemma 2. Since

$$(a+s+x)A_x = aA_{x-1} + (x+1)A_{x+1}, \text{ and } (a+s+x)B_x = aB_{x-1} + (x+1)B_{x+1},$$

by multiplying the first equation by B_x and the second one by A_x and next subtracting the two equations, we deduce a relation for U_x:

$$(x+1)U_{x+1} = aU_x \text{ for } x \geq 0.$$

Therefore, we have $U_x = \frac{a^x}{x!} U_0$ for $x \geq 0$. Since $A_{-1} = 0$ and $A_0 = 1$, we have $U_0 = B_{-1}$, which leads to

$$U_x = \frac{e^a a^{x+s-1}}{x! \Gamma(s)} \text{ for } x \geq 0. \tag{30}$$

In Theorem 2, we solve the balance equations using the building blocks found with Lemma 2. As for A_x and B_x, we extend the definition of E_x and F_x to $x = -1$, with $E_{-1} = \frac{a^{s-1}}{\Gamma(s)}$ and $F_{-1} = 0$. To determine the stationary probabilities, we also need the asymptotic expressions of A_x and B_x as x grows large as provided in Lemma 3. We write $u_x \underset{x\to x_0}{\sim} v_x$ to indicate that $\lim_{x\to x_0} \frac{u_x}{v_x} = 1$ for $x_0 \in \mathbb{R}$.

Lemma 3. *The asymptotic expressions of A_x and B_x as x grows large are given by*

$$A_x \underset{x\to\infty}{\sim} \frac{e^a x^{s-1}}{\Gamma(s)}, \text{ and } B_x \underset{x\to\infty}{\sim} \frac{a^{x+s}}{\Gamma(x+1+s)}.$$

Moreover, $\lim_{x\to\infty} \frac{A_x}{B_x} = \infty$.

Theorem 2. *For Policy π_0, we obtain*

$$q_0^0 = \left[\sum_{x=0}^{n}\left(F_x + \frac{s}{a}\frac{1+D_n}{C_n}C_x - \frac{s}{a}D_x\right) + \frac{F_n}{E_n}\sum_{x=n+1}^{\infty}E_x\right]^{-1}, q_x^0 = q_0^0 F_x \text{ for } 0 \le x \le n,$$
$$q_x^0 = q_0^0\frac{E_x F_n}{E_n} \text{ for } x \ge n, \text{ and } p_x^0 = q_0^0\frac{s}{a}\left(C_x\frac{1+D_n}{C_n} - D_x\right) \text{ for } 0 \le x \le n.$$

For Policy π_1, we get

$$q_0^1 = \left[\frac{A_{n+1}}{C_{n+1}}\sum_{x=0}^{n}C_x + \frac{A_n}{E_n}\sum_{x=n+1}^{\infty}E_x\right]^{-1}, q_x^1 = q_0^1 A_x \text{ for } 0 \le x \le n, q_x^1 = q_0^1 A_n\frac{E_x}{E_n} \text{ for } x \ge n,$$
$$\text{and } p_x^1 = q_0^1\left(\frac{A_{n+1}}{C_{n+1}}C_x - A_x\right) \text{ for } 0 \le x \le n.$$

For Policy π_2, we have

$$p_0^2 = \left[\sum_{x=-1}^{n-2}\frac{E_x}{E_{-1}} + \frac{a\frac{E_{n-2}}{E_{-1}}}{a+s+n-1-n\frac{B_n}{B_{n-1}}}\sum_{x=n-1}^{\infty}\frac{C_x}{C_{n-1}}\right]^{-1}, q_x^2 = p_0^2\frac{E_x}{E_{-1}} \text{ for } 0 \le x \le n-2,$$
$$q_x^2 = p_0^2\frac{a\frac{E_{n-2}}{E_{-1}}}{a+s+n-1-n\frac{B_n}{B_{n-1}}}\frac{B_x}{B_{n-1}} \text{ for } x \ge n-1,$$
$$\text{and } p_x^2 = p_0^2\frac{a\frac{E_{n-2}}{E_{-1}}}{a+s+n-1-n\frac{B_n}{B_{n-1}}}\left(\frac{C_x}{C_{n-1}} - \frac{B_x}{B_{n-1}}\right) \text{ for } x \ge n.$$

From the stationary probabilities, we deduce the agent's occupation rate under Policy π_i, termed p_B^i, and the agent's idling probability, termed p_I^i, as $p_B^i = \sum_{x=0}^{\infty} q_x^i = 1 - p_I^i$ for $i = 0, 1, 2$. The expected congestion cost function under Policy π_i, termed $E(f_S(N))^i$, is computed as $E(f_S(N))^i = \sum_{x=0}^{\infty} f_B(x)q_x^i + f_I(x)p_x^i$ for $i = 0, 1, 2$. For Policies π_0 and π_1, since the building blocks A_x, C_x, D_x, E_x and F_x can be directly computed, we may estimate the performance measures from the stationary probabilities. For Policy π_2, B_x remains to be computed as explained in Section 5.2. It should be noted that for p_B^2, only B_n and B_{n-1}

need to be determined. This can be explained from flow conservation, as

$$\lambda = \mu p_B^2 + \gamma \sum_{x=0}^{\infty} x(p_x^2 + q_x^2),$$

where $\sum_{x=0}^{\infty} x(p_x^2 + q_x^2)$ only depends on B_n and B_{n-1}.

5.2 Evaluation of B_x

In this section, we show how B_x can be computed. From (30), we could determine an iterative way to compute B_x using $B_x = \frac{A_x B_{x-1} - U_x}{A_{x-1}}$. However, this cannot be done due to $A_{-1} = 0$. Instead, in Proposition 1, we show how B_x can be computed as a function of B_0. Next, using the asymptotic expression of B_x as x grows large, we estimate the value of B_0.

Proposition 1. *For $x \geq -1$, we have*

$$B_{x+1} = A_{x+1} B_0 - \frac{e^a a^{s-1}}{\Gamma(s)} \frac{a}{(x+1)!} \sum_{k=0}^{x} a^k h_{k,x}(s), \tag{31}$$

where $h_{k,x}(s) = \sum_{i=0}^{x-k} \alpha_{i,k,x} s^i$ is a polynomial in s of degree $x-k$ that does not depend on a, with $\alpha_{0,0,0} = 1$, $\alpha_{0,0,1} = \alpha_{1,0,1} = \alpha_{0,1,1} = 1$, and furthermore

$$\begin{aligned} &\alpha_{0,k,x} = x\alpha_{0,k,x-1} + \alpha_{0,k-1,x-1} - x\alpha_{0,k-1,x-2}, \\ &\alpha_{i,k,x} = \alpha_{i-1,k,x-1} + x\alpha_{i,k,x-1} + \alpha_{i,k-1,x-1} - x\alpha_{i,k-1,x-2}, \text{ for } 1 \leq i \leq x-k-1, \\ &\alpha_{x-k,k,x} = \alpha_{x-k-1,k,x-1} + \alpha_{x-k,k-1,x-1} \text{ for } x \geq 2, \end{aligned} \tag{32}$$

with $h_{x,x}(s) = 1$, $h_{0,x} = \frac{\Gamma(s+x+1)}{\Gamma(s+1)}$ for $x \geq 0$, and $h_{k,x}(s) = (s+x)h_{k,x-1}(s) + h_{k-1,x-1}(s) - xh_{k-1,x-2}(s)$ for $1 \leq k \leq x-1$ and $x \geq 2$.

Using Proposition 1, we approximate the value of B_0 and next deduce an approximate value for B_x for $x \geq 0$. Using Proposition 1, we have

$$B_0 = \frac{B_{m+1}}{A_{m+1}} + \frac{e^a a^{s-1}}{\Gamma(s)} \frac{\frac{a}{(m+1)!} \sum_{k=0}^{m} a^k h_{k,x}(s)}{A_{m+1}},$$

for $m > 0$. From Lemma 3 we have $\lim_{m\to\infty} \frac{B_{m+1}}{A_{m+1}} = 0$. Therefore, we approximate B_0 using

$$B_0 \underset{m\to\infty}{\sim} \frac{e^a a^{s-1}}{\Gamma(s)} \frac{\frac{a}{(m+1)!}\sum_{k=0}^{m} a^k f_{k,m}(s)}{A_{m+1}},$$

for a large m. From Proposition 1, we thus obtain an iterative method to compute B_x.

The method in Proposition 1 provides an explicit expression for $B_x - A_x B_0$. However, the computation of the coefficients $\alpha_{i,k,x}$ can be long as we need to select m sufficiently high to obtain a good approximation for B_0. To avoid this difficulty, in Proposition 2, we propose an explicit expression of B_x as a function of A_y for $y \geq 0$. This proposition allows us to derive B_x numerically.

Proposition 2. *For $x \geq -1$, we have*

$$B_x = \frac{e^a a^{s-1}}{\Gamma(s)} A_x \sum_{k=x+1}^{\infty} \frac{a^k}{k! A_k A_{k-1}}. \tag{33}$$

Thus, Proposition 2 determines an explicit way to compute a complex integral that could not be derived explicitly.

5.3 Idling duration

In this section, we determine the expected duration of the idling period for the agent, termed $E(I)^i$, under Policy π_i for $i = 0, 1, 2$. To this end, we analyze the Laplace transform of the density function of the first passage time from a given state (x, I) to state *end of the idling period*, termed T_x^i, in the variable $t \geq 0$ with Policy π_i for $i = 0, 1, 2$. During an idling period, only arrivals and abandonments from the queue may occur. With Policies π_0 and π_1, the agent idles for states where $0 \leq x \leq n$. The idling period can only be interrupted after a $\lambda-$transition from state (n, I). With Policy π_2, the agent idles for states $x \geq n$ and the idling period can only be interrupted after an $n\gamma-$transition from state (n, I). Therefore, T_x^i is solution of

$$T_x^i(a + x + \theta) = aT_{x+1}^i + xT_{x-1}^i \text{ for } 0 \leq x \leq n \text{ and } i = 0, 1, \text{ with } T_{n+1} = 1 \text{ and} \tag{34}$$

$$T_x^2(a + x + \theta) = aT_{x+1}^2 + xT_{x-1}^2 \text{ for } x \geq n, \text{ with } T_{n-1} = 1,$$

with $\theta = \frac{t}{\gamma}$. In Proposition 3, we solve (34) and provide the expression of T_x^i for $i = 0, 1, 2$ as a function the confluent hypergeometric functions of the first and second kind (Daalhuis, 2010), defined by

$$M_1(a,b,c) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{cu} u^{a-1}(1-u)^{b-a-1} \mathrm{d}u, \text{ and}$$
$$M_2(a,b,c) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-cu} u^{a-1}(1+u)^{b-a-1} \mathrm{d}u, \text{ with } b \geq a > 0.$$

Proposition 3. *The solution of (34) is given by*

$$T_x^i = \frac{M_2(\theta, x+\theta+1, a)}{M_2(\theta, n+\theta+2, a)} \textit{ for } i = 0, 1, \textit{ and} \tag{35}$$
$$T_x^2 = \frac{x!}{(n-1)!} \frac{\Gamma(n+\theta)}{\Gamma(x+1+\theta)} \frac{M_1(\theta, x+\theta+1, a)}{M_1(\theta, n+\theta, a)}. \tag{36}$$

We observe that for the three policies, we have $\lim_{\theta \to 0} T_x^i = 1$, which indicates that the idling period ends with probability 1. We next deduce the expected idling time starting at state x, termed $E(I_x)^i$, through $E(I_x)^i = -\frac{1}{\gamma}\frac{\partial T_x^i}{\partial \theta}|_{\theta=0}$, for $i = 0, 1, 2$. For Policies π_0 and π_1, $E(I_x)^i$ can be derived explicitly. We obtain, after some algebra that

$$E(I_x)^i = \frac{1}{\lambda} \sum_{y=x}^{n} \sum_{k=0}^{y} \left(\frac{\gamma}{\lambda}\right)^k \frac{y!}{(y-k)!} \text{ for } 0 \leq x \leq n \text{ and } i = 0, 1. \tag{37}$$

For Policy π_2, we cannot obtain an explicit expression for $E(I_x)^2$. We instead need to compute the derivative of T_x^2 in θ numerically.

We now express $E(I)^i$ using $E(I_x)^i$ and the stationary probabilities found in Section 5.1. For Policy π_0, the idling period starts from state $x = 0$ with probability 1. Therefore, the expected idling duration is $E(I_0)^0$. For Policy π_1, the idling period starts from state x with probability $\frac{q_x^1}{\sum_{k=0}^{n} q_x^1}$ with expected duration $E(I_x)^1$ for $0 \leq x \leq n$. Finally for Policy π_2, either the idling period starts from state $(0, I)$ with an expected duration of $\frac{1}{\lambda}$ and probability $\frac{q_0^2}{q_0^2 + \sum_{x=n}^{\infty} q_x^2}$ or it starts from state (x, I) for $x \geq n$ with probability $\frac{q_x^2}{q_0^2 + \sum_{x=n}^{\infty} q_x^2}$ and expected duration $E(I_x)^2$. Therefore, the expected duration of the idling period is computed with

$$E(I)^0 = E(I_0)^0, \quad E(I)^1 = \frac{\sum_{x=0}^{n} q_x^1 E(I_x)^1}{\sum_{x=0}^{n} q_x^1}, \text{ and } E(I)^2 = \frac{\frac{1}{\lambda} q_0^0 + \sum_{x=n}^{\infty} q_x^2 E(I_x)^2}{q_0^2 + \sum_{x=n}^{\infty} q_x^2}.$$

6 Numerical analysis

In this section, we conduct a numerical analysis based on the results of Sections 4 and 5. In Section 6.1, we show how the optimal threshold n can be computed for each policy and provide examples where either Policy π_1 or Policy π_2 is optimal. Next, in Section 6.2, we show how the busy probability, p_B^i, can be approximated for $i = 1, 2$ when either the service speed tends to infinity, the arrival rate tends to infinity, or both the arrival rate and threshold level tend to infinity, leading to a Normal approximation. Finally, in Section 6.3, we compare the optimal idling policies with a rejection policy, where some customers are rejected upon arrival to reduce the system's congestion and increase the agent's idling probability.

6.1 Policy evaluation

First we explain how the optimal threshold level should be computed, and next we illustrate optimization problems where either Policy π_1 or Policy π_2 is optimal. For the numerical experiments, we select $\mu = 0.5$, $\lambda = 3$, and $\gamma = 1$. With these parameters, the interval of possible values for the occupation rate is large as without implementing an idling policy, we find $p_B = 98.38\%$, while with $n = 0$ with Policy π_2, or by letting n tend to infinity with Policies π_0 and π_1, we obtain $p_B = 0\%$.

Computation of the optimal threshold level. To compare Policies π_0, π_1 and π_2, we need to determine the optimal threshold level n to solve Problem 1. The complexity of the stationary probabilities in Theorem 2 does not allow us to prove the effect of increasing the threshold level n on the occupation rate and congestion cost function for each policy. Intuitively, by increasing n for Policies π_0 and π_1 (for Policy π_2), we increase (decrease) the number of states where the agent must idle while some customers are waiting in the queue. Therefore, the occupation rate should decrease in n for Policies π_0 and π_1 and increase in n for Policy π_2. This intuition is confirmed by numerical evaluations as illustrated in Figure 2(a). Based on this intuition, Algorithms 1 and 2 provide a simple way to determine the optimal threshold level, n^*, to solve Problem 1. Since $\lim_{n\to\infty} p_B^i = 0$ for Policy π_i for $i = 0, 1$ and we may find situations where $\lim_{n\to\infty} p_B^2 < \overline{p_B}$, Algorithms 1 and 2 may not stop. Therefore, we introduce an upper bound for the search of n, termed $\overline{n}$. It should also be noted that since $p_B^2 = 0$ for $n = 0$, we do not need to specify the value of p_B^2 at the first step of Algorithm 2. Finally in the algorithms, the threshold n is assumed to be an integer. Therefore, at each iteration, the threshold is increased by one. We could instead assume that the threshold is a positive real and increase the threshold by a quantity that is lower than one at each iteration. This would provide an improved solution to Problem 1. For the computation of the performance measures, it

Algorithm 1 Computation of the optimal threshold level n^* for Policy π_i for $i = 0, 1$

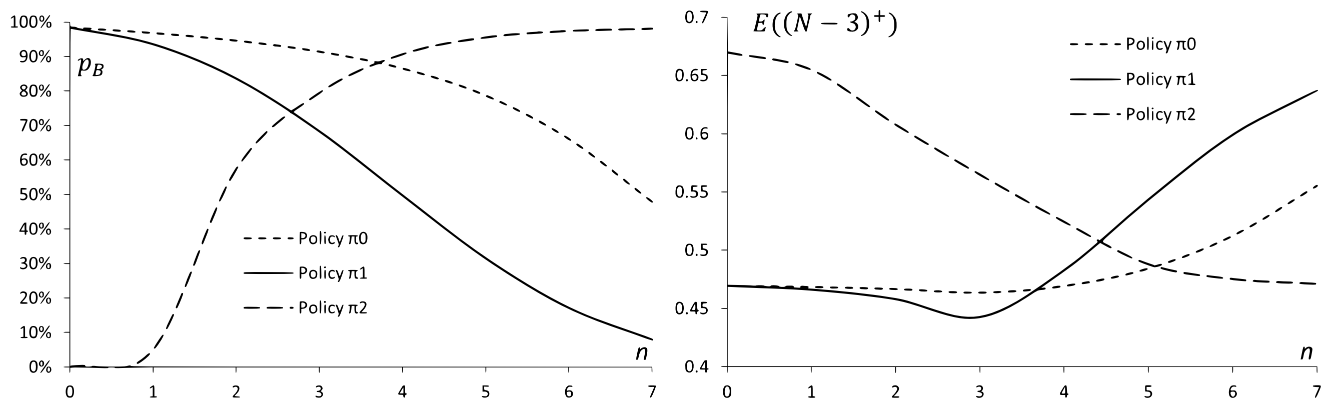
- Step 1: Set $n = n^* = 0$ and compute p_B^i and $E(f_S(N))^i$. If $p_B^i \leq \overline{p_B}$, then set $Z = E(f_S(N))^i$ and move to step 2. If $p_B^i > \overline{p_B}$, then move to step 3.
- Step 2: Increase n by 1 and compute $E(f_S(N))^i$.
  - If $E(f_S(N))^i < Z$, then set $n^* = n$ and $Z = E(f_S(N))^i$ and go back to step 2.
  - If $E(f_S(N))^i \geq Z$, then go back to step 2 directly.
- Step 3: Increase n by 1 and compute p_B^i and $E(f_S(N))^i$.
  - If $p_B^i \leq \overline{p_B}$, then set $Z = E(f_S(N))^i$ and move to step 2.
  - If $p_B^i > \overline{p_B}$, then go back to step 3.
- Stopping criterion: $n = \overline{n}$.

Algorithm 2 Computation of the optimal threshold level n^* for Policy π_2

- Step 1: Set $n = n^* = 0$ and compute $E(f_S(N))^2$. Set $Z = E(f_S(N))^2$ and move to step 2.
- Step 2: Increase n by 1 and compute p_B^2 and $E(f_S(N))^2$. If $p_B^2 > \overline{p_B}$, then stop the algorithm. Otherwise, if $p_B^2 \leq \overline{p_B}$ then
  - If $E(f_S(N))^2 < Z$, then set $n^* = n$ and $Z = E(f_S(N))^2$, and go back to Step 2.
  - If $E(f_S(N))^2 \geq Z$, then go back to step 2 directly.
- Stopping criterion: $n = \overline{n}$.

means that we employ the equivalent continued functions in n for the stationary probabilities (Jagers and Van Doorn, 1986). In practice, a noninteger value for the threshold n can be obtained by randomizing in between two adjacent integer threshold policies as explained in Bhulai et al. (2012).

A simplification could be obtained for Algorithms 1 and 2 by assuming that the congestion cost function is also monotonous in n. We would expect that congestion-related performance measures would have opposite monotonicity properties in n; that is, they would be increasing in n for Policies π_0 and π_1 and decreasing in n for Policy π_2. We observe that this is not necessarily the case. For instance, in Figure 2(b), we observe that $E(N-3)^+$ is decreasing in n for Policy π_1 for $0 \leq n \leq 3$. However, these cases only occur as exceptions. For $E((N-3)^+)$, we count the number of customers in the queue above 3. It is then advisable to select $n = 3$ instead of $n < 3$. In this way, the agent has the highest chance of being available when the number of customers in the queue reaches 3. However, for most congestion-related performance

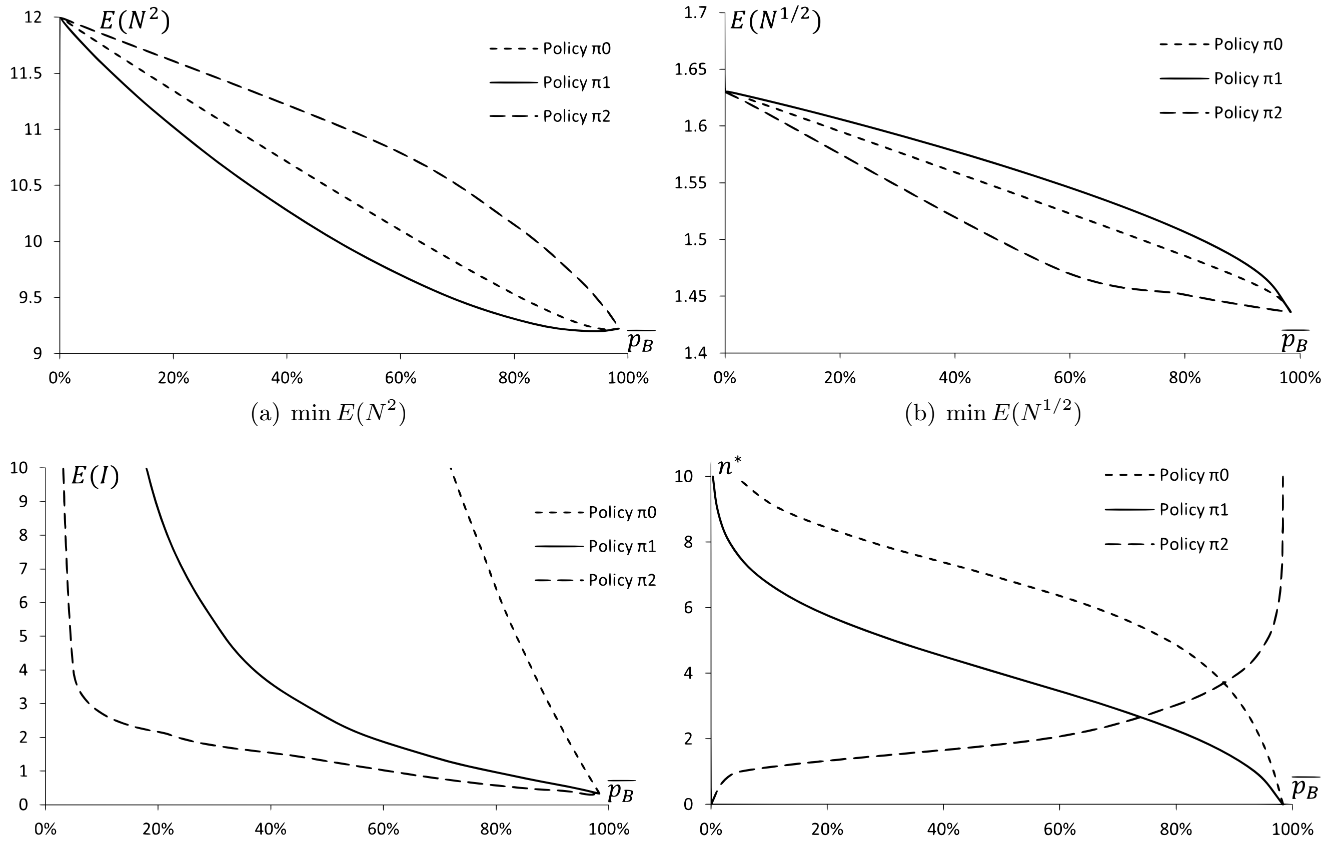


(a) p_B (b) $E((N-3)^+)$

Figure 2: Numerical illustration, effect of n ($\mu = 0.5$, $\lambda = 3$, $\gamma = 1$)

measures, such as $E(N)$, $E(W)$ or $E(N^k)$ for $k \geq 0$, we observe that the congestion performance measure evolves in the opposite direction as the occupation rate. Even for counterexamples such as $E((N-3)^+)$ in Figure 2(b), we observe that the expected monotonicity property in the threshold n holds in most cases. The intuition is that with an idling policy, decreasing the occupation rate reduces the departure rate of the system as the agent is not working in some states, which induces more congestion. In the numerous cases where the monotonicity of the congestion cost function in n can be shown, the solution to Problem 1 can simply be obtained by increasing n until the constraint on the occupation rate is saturated.

Policy comparison. In Figure 3, we consider two examples of optimization problem, where either the second moment $E(N^2)$, or half moment $E(N^{1/2})$ of the number of customers in the queue is minimized. The second (half) moment is a convex (concave) function in the number of customers in the queue that illustrates Theorem 1 when the cost function belongs to $\mathcal{C}_1$ ($\mathcal{C}_3$). The moments $E(N^2)$ and $E(N^{1/2})$ are monotonous in n for each idling policy. Therefore, as mentioned above, we obtain the optimal threshold level n^* by saturating the constraint in Problem 1 (i.e., $p_B^i = \overline{p_B}$ for $i = 0, 1, 2$). In Figures 3(a) and 3(b), we provide the optimal value for $E(N^2)$ and $E(N^{1/2})$ as functions of $\overline{p_B}$ (i.e., the solution to Problem 1). In Figure 3(c), we give the corresponding value of the expected idling time duration. Finally in Figure 3(d), we specify the optimal threshold level for each policy. It should be noted that since n^* is selected to saturate the constraint on the occupation rate, for each value of $\overline{p_B}$, there is a unique optimal threshold n^*, which is independent of the objective to either minimize $E(N^2)$ or $E(N^{1/2})$. Consequently, for each value of $\overline{p_B}$, there is also a unique value of the expected idling time that does not depend on the objective to either minimize $E(N^2)$ or $E(N^{1/2})$.



(a) $\min E(N^2)$ (b) $\min E(N^{1/2})$

(c) Evaluation of $E(I)$ (d) Optimal threshold level n^*

Figure 3: Numerical comparison ($\mu = 0.5$, $\lambda = 3$, $\gamma = 1$)

As expected from Theorem 1, Policy π_1 is optimal for convex performance measures (Figure 3(a)), while Policy π_2 is optimal for concave ones (Figure 3(b)). In addition, Figure 3(c) shows that Policy π_0 provides the longest duration of the idling period. This can be interesting if the agent wishes to avoid having too many switches from idle to busy. The difference between the different policies is the highest for intermediate values of $\overline{p_B}$. When $\overline{p_B}$ is close to 0% or 100%, the agent either always or never idles; therefore, the idling policy does not impact the system's performance. We further observe that the congestion-related performance measure is convex in $\overline{p_B}$ for the optimal policy. This argues in favor of employing an idling policy, because a reduction of the objective occupation rate from a work-conserving situation will only result in a small deterioration of the congestion-related performance measure.

6.2 Approximations

In most cases, the congestion-related performance measure is monotonous in n. Therefore, the optimal threshold level n^* is determined by the constraint on the occupation rate in Problem 1. To facilitate the computation of the optimal threshold level n^* and provide a better understanding of the complex formulas in Theorem 2, we approximate the occupation rate for different asymptotic cases. In this section, we only focus on Policies π_1 and π_2 because only one of these two policies is optimal. We consider a fast-agent approximation where the service speed tends to infinity, a highly congested system where the arrival rate tends to infinity, and a system where both a and n tend to infinity with respect to $\frac{n-a}{\sqrt{a}} = \beta$.

6.2.1 Fast-agent approximation

We first consider the asymptotic case where s tends to infinity while a and n are held constant. This case corresponds to situations where the abandonment and arrival rates are significantly lower than the service rate. In Proposition 4, we provide the asymptotic expressions of p_B^1 and p_B^2 as s tends to infinity. These asymptotic expressions can be used as approximations of the occupation rate. It should be noted that in the case where $n \geq 3$, we can further approximate p_B^2 with $p_B^2 \underset{s\to\infty}{\sim} \frac{a}{s}$.

Proposition 4. *For $n \geq 0$, the asymptotic expressions of p_B^1 and p_B^2 as s tends to infinity are given by*

$$p_B^1 \underset{s\to\infty}{\sim} \frac{a^{n+1}}{sn!\left(\sum\limits_{k=0}^{n} \frac{a^x}{x!}\right)} \text{ and } p_B^2 \underset{s\to\infty}{\sim} \frac{\frac{a}{s} + \left(\frac{a}{s}\right)^2 + \frac{a\left(\frac{a}{s}\right)^{n-2}}{a+n-1}}{1 + \frac{a}{s} + \frac{a\frac{(n-1)!}{s^{n-1}} \sum\limits_{x=n-1}^{\infty} \frac{a^x}{x!}}{a+n-1}}. \tag{38}$$

In Table 3, we show how the fast-agent approximations become close to the exact value of p_B^i as s increases for $i = 1, 2$. The relative difference between the approximation and the exact result is computed as $\frac{p_B^{\text{Exact}} - p_B^{\text{Approximated}}}{p_B^{\text{Approximated}}}$. As expected, we observe that the quality of the fast-agent approximation improves with s.

Table 3: Fast-agent approximation, p_B $(a = 4, n = 5)$

s	Exact values		Approximation		Relative difference	
	Policy π_1	Policy π_2	Policy π_1	Policy π_2	Policy π_1	Policy π_2
5	13.74%	63.45%	15.92%	70.85%	-13.69%	-10.44%
10	7.52%	37.89%	7.97%	41.19%	-5.59%	-8.03%
20	3.91%	19.77%	3.99%	20.29%	-1.94%	-2.60%
50	1.59%	7.99%	1.60%	8.02%	-0.31%	-0.46%
100	0.80%	4.00%	0.80%	4.00%	-0.13%	-0.12%

6.2.2 High-workload approximation

We now consider the case where a tends to infinity and both n and s are held constant. This case corresponds to a congested system where the arrival rate is greater than the service and abandonment rates. In Proposition 5, we provide the asymptotic expressions of p_B^1 and p_B^2 for large a. It should be noted that the asymptotic expression for p_B^2 is valid only $n \geq 2$ because the terms in E_x can only be neglected if $n \geq 2$. For $n = 0$, we have $p_B^2 = 0$ and for $n = 1$ we find that $\lim_{a \to \infty} p_B^2 = 0$. The asymptotic expression of p_B^2 reveals that the occupation rate does not necessarily increase in a for Policy π_2. The reason is that as a increases, while n is held constant, the probability to exceed n customers in the queue increases, leading the agent to idle more.

Proposition 5. *The asymptotic expressions of p_B^1 and p_B^2 as a tends to infinity are given by*

$$p_B^1 \underset{a \to \infty}{\sim} 1 - \frac{\frac{s}{a}}{1 + \frac{e^a \Gamma(n+1+s)}{a^{n+s}}} \text{ for } n \geq 0 \text{ and } p_B^2 \underset{a \to \infty}{\sim} e^{-a} \frac{a^n}{s(n-2)!}, \text{ for } n \geq 2. \quad (39)$$

In Table 4, we illustrate how the high-workload approximation becomes close to the exact value for the occupation rate as a increases. As in Table 3, we derive the relative difference between the exact and approximated results.

Table 4: High-workload approximation, p_B $(s = 4, n = 5)$

a	Exact values		Approximation		Relative difference	
	Policy π_1	Policy π_2	Policy π_1	Policy π_2	Policy π_1	Policy π_2
8	63.80%	32.33%	94.48%	45.80%	-32.47%	-29.40%
10	80.96%	15.92%	95.55%	18.92%	-15.27%	-15.82%
15	97.92%	0.89%	99.16%	0.97%	-1.25%	-8.55%
30	100.00%	0.00%	100.00%	0.00%	0.00%	-0.02%
50	100.00%	0.00%	100.00%	0.00%	0.00%	0.00%

6.2.3 Normal approximation

We now propose a Normal approximation for the occupation rate. To this end, we assume that s is fixed and that both n and a tend to infinity with $\frac{n-a}{\sqrt{a}} = \beta$. This approach relates the computation of the optimal threshold level to a square-root staffing rule (e.g., see Whitt (2007)) and the performance measures to the asymptotic Halfin-Whitt regime (e.g., see Whitt (1974); Reed (2009); Braverman (2020)). In Proposition 6, we provide the asymptotic expressions of p_B^i for $i = 1, 2$. We introduce the cumulative distribution function (cdf) of a Normal distribution with mean 0 and standard deviation 1, $\Phi(x)$ for $x \in \mathbb{R}$, and the parabolic cylinder function of index x and argument y, $P_x^c(y)$ for $x, y \in \mathbb{R}$ (Temme, 2010). Recall that $P_x^c(y)$ can be computed as $P_x^c(y) = \Lambda\left(-x - \frac{1}{2}, y\right)$, where

$$\Lambda(t,z) = \frac{\sqrt{\pi} e^{-\frac{z^2}{4}} \sum_{k=0}^{\infty} \frac{z^{2k} \prod_{i=0}^{k-1}\left(t+\frac{1}{2}+2i\right)}{(2k)!}}{2^{\frac{t}{2}+\frac{1}{4}}\Gamma\left(\frac{3}{4}+\frac{t}{2}\right)} - \frac{\sqrt{\pi} e^{-\frac{z^2}{4}} \sum_{k=0}^{\infty} \frac{z^{2k+1} \prod_{i=0}^{k-1}\left(t+\frac{3}{2}+2i\right)}{(2k+1)!}}{2^{\frac{t}{2}-\frac{1}{4}}\Gamma\left(\frac{1}{4}+\frac{t}{2}\right)},$$

for $t, z \in \mathbb{R}$.

Proposition 6. *The asymptotic expressions of p_B^1 and p_B^2 as a and n tend to infinity with $\frac{n-a}{\sqrt{a}} = \beta$ are given by*

$$p_B^1 \underset{a,n\to\infty}{\sim} 1 - \Phi(\beta) + \frac{\Phi'(\beta)}{s}\left(\beta + \frac{P_{1-s}^c(-\beta)}{P_{-s}^c(-\beta)}\right) \text{ and } p_B^2 \underset{a,n\to\infty}{\sim} \Phi(\beta).$$

In Table 5, we evaluate the relative difference between the approximations presented in Proposition 6 and the exact results for p_B^1 and p_B^2 for $a = 100$, $a = 1,600$, and $a = 3,600$. As the approximations in Proposition 6 are followed by a term in $\frac{1}{\sqrt{a}}$, the convergence is slow. This is particularly the case for Policy π_1. For this policy, we indicate an improved approximation of p_B^1, which involves the term in $\frac{1}{\sqrt{a}}$. This improved approximation is closer to the exact value of p_B^1. It is given by

$$\begin{aligned} p_B^1 \underset{a,n\to\infty}{\sim} & 1 - \Phi(\beta) + \frac{\Phi'(\beta)}{s}\left(\beta + \frac{P_{1-s}^c(-\beta)}{P_{-s}^c(-\beta)}\right) \\ & + \frac{1}{\sqrt{a}}\left(\Phi(\beta) - \frac{\Phi'(\beta)}{s}\left(\beta + \frac{P_{1-s}^c(-\beta)}{P_{-s}^c(-\beta)}\right)\right)\left((1-\Phi(\beta))\left(\beta(s-1) - \frac{P_{1-s}^c(-\beta)}{P_{-s}^c(-\beta)}\right) - s\Phi'(\beta)\right) - \frac{\Phi'(\beta)}{2\sqrt{a}}. \end{aligned}$$

This expression can be deduced from the proof of Proposition 6.

Table 5: Normal approximation, p_B ($s = 3$)

	Policy π_1						
β	Approximation	Exact values $a = 100$	$a = 1,600$	$a = 3,600$	Relative difference $a = 100$	$a = 1,600$	$a = 3,600$
-0.5	90.05%	77.67%	83.64%	84.24%	-13.75%	-7.12%	-6.44%
-0.2	78.38%	68.37%	75.36%	76.11%	-12.77%	-3.86%	-2.91%
0	71.22%	61.45%	68.79%	69.60%	-13.72%	-3.41%	-2.27%
0.2	64.99%	54.17%	61.56%	62.40%	-16.65%	-5.28%	-3.98%
0.5	58.60%	43.12%	49.99%	50.81%	-26.43%	-14.69%	-13.30%
β	Approximation	Improved approximation $a = 100$	$a = 1,600$	$a = 3,600$	Relative difference $a = 100$	$a = 1,600$	$a = 3,600$
-0.5	90.05%	81.63%	88.52%	89.06%	-4.85%	-5.52%	-5.41%
-0.2	78.38%	68.71%	76.44%	77.11%	-0.49%	-1.41%	-1.31%
0	71.22%	61.56%	69.19%	69.89%	-0.18%	-0.57%	-0.41%
0.2	64.99%	55.48%	62.93%	63.64%	-2.36%	-2.17%	-1.94%
0.5	58.60%	48.87%	54.00%	53.18%	-11.77%	-7.42%	-4.46%
	Policy π_2						
β	Approximation	Exact values $a = 100$	$a = 1,600$	$a = 3,600$	Relative difference $a = 100$	$a = 1,600$	$a = 3,600$
-0.5	30.85%	32.14%	30.93%	30.85%	4.16%	0.24%	0.00%
-0.2	42.07%	44.91%	42.27%	42.11%	6.74%	0.47%	0.08%
0	50.00%	53.73%	50.13%	50.07%	7.47%	0.25%	0.14%
0.2	57.93%	61.81%	58.19%	58.01%	6.70%	0.46%	0.14%
0.5	69.15%	73.06%	69.37%	69.16%	5.66%	0.33%	0.02%

6.3 Comparison with customer rejection

Another way to preserve the agent's idling time is to reject some customers upon arrival to avoid having a too highly congested system. Rejection policies were proven to be optimal for optimization problems that involve a trade-off between congestion and rate of rejected customers (Koole, 2007; Koçağa and Ward, 2010). Therefore, we are interested in comparing the optimal idling policies of the current paper with a rejection policy with the aim of solving Problem 1. We consider a rejection policy, termed Policy π_r controlled by a rejection threshold, n, on the number of customers in the queue such that a customer is rejected upon arrival if the queue size is n. With Policy π_r, the performance measures are obtained from the analysis of the M/M/1+M/n+1 queue.

In Table 6 we provide the solution to Problem 1 for Policy π_r and the optimal idling policy for different values of the objective occupation rate, $\overline{p_B}$, with the aim of minimizing either $E(N^2)$ or $E(N^{1/2})$ as in Figure 3. We also compute the relative difference in congestion cost, RD, computed as the difference between the congestion cost with Policy π_i and the one with Policy π_r divided by the congestion cost with Policy π_r for $i = 1, 2$. With Policy π_r, the interval of achievable values for $\overline{p_B}$ is more restricted than for the idling policies (see, Figure 3). In this case with Policy π_r, the lowest achievable value for $\overline{p_B}$ is 85.71%. To make the comparison in Table 6, we only present values of $\overline{p_B}$ where Problem 1 has solutions with Policy π_r.

Table 6: Comparison with customers' rejection ($a = 3$, $s = 0.5$, $\gamma = 1$)

	$E(N^2)$			$E(N^{1/2})$		
$\overline{p_B}$	Policy π_1	Policy π_r	RD	Policy π_2	Policy π_r	RD
85.71%	9.238	0.000	-	1.446	0.000	-
94.74%	9.237	0.632	1362.53%	1.439	0.632	127.80%
97.01%	9.230	2.084	342.93%	1.437	0.969	48.30%
97.81%	9.225	3.950	133.55%	1.436	1.175	22.26%
98.15%	9.223	5.787	59.37%	1.436	1.301	10.42%
98.29%	9.221	7.262	26.98%	1.436	1.373	4.63%
98.35%	9.221	8.246	11.82%	1.436	1.409	1.89%
98.37%	9.221	8.796	4.83%	1.436	1.426	0.70%
98.38%	9.221	9.058	1.80%	1.436	1.433	0.23%

We observe that Policy π_r leads to lower values for $E(N^2)$ and $E(^{1/2})$ than idling policies. The reason is that Policy π_r is work-conserving. With Policy π_1 or Policy π_2, some customers wait in the system while the agent is not working. This deteriorates the operational performance measures. Moreover, we observe that $E(N^2)$ and $E(N^{1/2})$ are increasing in $\overline{p_B}$ for Policy π_r, while the opposite is true for Policies π_1 and π_2. For Policy π_r, rejecting customers simultaneously reduces the system's congestion and the agent's occupation rate, while for the idling policies, increasing the idling probability does not reduce the flow of arriving customers while reducing the system's departure rate, which consequently increases the system's congestion. It should also be mentioned that Policy π_r leads to shorter breaks for the agent than idling policies. Specifically, the expected idling duration is $\frac{1}{\lambda}$ with Policy π_r, which is lower than the expected idling durations found in Section 5.3. In practice, the comparison reveals that idling and rejection policies should be implemented in different contexts. When the objective occupation rate is low, only idling policies can be implemented as rejection policies cannot reach low occupation rate values. When both policies can be implemented, rejection policies outperform idling ones but under the cost of rejecting some customers while idling policies allow all customers to join the service.

7 Conclusion

We studied a single-agent queue with abandonment where the agent has the possibility to idle when customers are waiting in the queue. We aimed to determine the optimal policy to minimize congestion-related performance measures while maintaining the occupation rate of the agent below a certain threshold. To this end, we defined three classes of function that capture how the congestion cost function could behave in the number of customers and state of the agent. Using a Markov decision process, we proved, under some conditions on the system parameters that it is optimal to idle either below or above a threshold on the queue length. We next evaluated the performance measures under the two threshold policies and under a reference policy where the agent only takes breaks when the system is empty. The evaluation involved complex integrals that we derived numerically. We also determined the Laplace transform of the idling duration, which we expressed in terms of confluent hypergeometric functions. In addition, we provided asymptotic expressions of the performance measures for fast-agent and heavy-traffic cases. Our numerical investigations showed that increasing the idling duration can also be beneficial for operational performance in some cases. Furthermore, the optimal idling policy provides significant improvement as compared to the reference policy when the system is highly congested and the desired occupation rate is close to 50%. We also showed that idling policies should be implemented instead of those involving customer rejection when the selected idling probability is high.

There are several avenues for future research. Although the discipline of service is first-come first-served, the expected wait at arrival may be significantly impacted by the selected idling policy. Therefore, it could be interesting to determine customers' behavior regarding their joining policy. Moreover, in this study we assume that service interruption is not permitted. For some applications, it could be possible to interrupt a service and put a customer back into the queue. This possibility could allow for achieving improved solutions for the optimization problem studied in this paper. The idling policies studied in this paper are quantity-based. This means that the idling decision is based on the number of customers present. It could be useful to compare these policies with delay-based ones where the agent instead makes idling decisions based on the wait experienced by some customers present in the system. The model definition can be extended in several directions. To reflect service system applications like in call centers or hospitals, it could be interesting to investigate the multi-agent case. However, the optimal policy becomes a complex state-dependent threshold policy that may not allow for determining the system's performance. Using more general assumptions for the service time, abandonment time, or arrival process may also be an interesting model extension to demonstrate the validity of the observations in wider contexts.

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A Table of notation

In Table 7, we recall the notations used throughout the paper.

Table 7: Table of notation

System state description	
x	Number of customers waiting in the queue with $x \in \mathbb{N}_0$
y	Status of the agent with $y \in \{I, B\}$
System parameters	
λ	Customers' arrival rate
μ	Service rate
γ	Abandonment rate
a	Ratio λ/γ
s	Ratio μ/γ
Random variables	
N	Number of customers waiting in the queue with $N \in \mathbb{N}_0$
S	Status of the agent with $S \in \{I, B\}$
$f_S(N)$	Congestion cost function
Markov decision process	
P	Lagrange multiplier
m	Upper bound for the queue size
$f_y(x)$	Congestion cost function at state (x, y)
$V_k(x, y)$	Value function over k steps at state (x, y)
$W_k(x)$	Minimizing operator at state x
$\mathcal{C}_i$ for $i = 1, 2, 3$	Set of functions for $f_y(x)$ where the optimal policy can be determined
Policies	
n	Control parameter of the idling policies (i.e., threshold on the number of customers in the queue)
n^*	Optimal threshold level of the idling policies
$\overline{n}$	Upper bound for the search of the optimal threshold level
Policy π_i for $i = 0, 1, 2$	Idling policies
Policy π_r	Rejection policy
p_x^i for $i = 0, 1, 2$	Stationary probability to have an idle agent and x customers in the queue with Policy π_i for $i = 0, 1, 2$
q_x^i for $i = 0, 1, 2$	Stationary probability to have a busy agent and x customers in the queue with Policy π_i for $i = 0, 1, 2$
$r^i_{(x,y),(x',y')}$ for $i = 0, 1, 2$	Transition rate from state (x, y) to state (x', y') for (x, y), $(x', y') \neq (x, y) \in \mathbb{N}_0 \times \{I, B\}$
Performance measures (the superscript i is used when Policy π_i is employed for $i = 0, 1, 2$)	
p_B	Proportion of time during which the agent is busy (occupation rate)
$\overline{p_B}$	Threshold level for the occupation rate
p_I	Proportion of idle time for the agent ($p_I = 1 - p_B$)
$E(f_S(N))$	Expected congestion cost
$E(W)$	Expected waiting time
T_x	Laplace transform in the variable t of the density function of the first passage time from state (x, I) to the end of the idling period (we also use the variable $\theta = \frac{t}{\gamma}$)
$E(I_x)$	Expected duration of the idling period starting at state (x, I)
$E(I)$	Expected duration of the idling period

B Proof of Lemma 1

Proof. To prove that it is optimal to idle below a certain threshold on the number of customers in the queue, we need to prove that if it is optimal to serve a customer in state x, then the same decision is optimal in state $x+1$. This implication is translated into if $V_k(x, I) \geq V_k(x-1, B)$, then $V_k(x+1, I) \geq V_k(x, B)$. This happens if

$$V_k(x+1, I) - V_k(x, B) \geq V_k(x, I) - V_k(x-1, B),$$

or equivalently,

$$V_k(x+1, I) + V_k(x-1, B) - V_k(x, B) - V_k(x, I) \geq 0.$$

In the same way, if $V_k(x, y)$ is such that

$$V_k(x+1, I) + V_k(x-1, B) - V_k(x, B) - V_k(x, I) \leq 0,$$

then it is optimal to idle above a certain threshold on the number of customers in the queue. □

C Proof of Theorem 1

Proof. We prove Theorem 1 by induction on k. In what follows, we recall the monotonicity properties that need to be proven for $V_k(x, y)$ and explain how the number of relations that need to be proven can be reduced. Next, we prove the induction step for the first order, second order and threshold properties. Note that for $k = 0$, all relations hold as $V_0(x, y) = 0$ for $0 \leq x \leq m$ and $y = I, B$.

C.1 Recall of the definitions of the properties

Consider a function $g_y(x)$ for $0 \leq x \leq m$ and $y = I, B$.

First order properties: For $g_y(x) \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3$, we have

$$\text{Increasing in } x\text{: } g_y(x+1) \geq g_y(x) \text{ for } 0 \leq x < m \text{ and } y = I, B, \text{ and} \tag{40}$$

$$\text{Effect of } y\text{: } g_B(x) \geq g_I(x) \text{ for } 0 \leq x \leq m. \tag{41}$$

Second order properties: For $g_y(x) \in \mathcal{C}_1 \cap \mathcal{C}_2$, we have

$$\text{Convexity in } (x, B)\text{: } g_B(x+1) + g_B(x-1) - 2g_B(x) \geq 0 \text{ for } 0 < x < m, \tag{42}$$

$$\text{Convexity in } (x, I)\text{: } g_I(x+1) + g_I(x-1) - 2g_I(x) \geq 0 \text{ for } 0 < x < m, \text{ and} \tag{43}$$

$$\text{Supermodularity: } g_B(x) + g_I(x-1) - g_B(x-1) - g_I(x) \geq 0 \text{ for } 0 < x \leq m. \tag{44}$$

For $g_y(x) \in \mathcal{C}_3$, we have

$$\text{Concavity in } (x, B)\text{: } g_B(x+1) + g_B(x-1) - 2g_B(x) \leq 0 \text{ for } 0 < x < m, \tag{45}$$

$$\text{Concavity in } (x, I)\text{: } g_I(x+1) + g_I(x-1) - 2g_I(x) \leq 0 \text{ for } 0 < x < m, \text{ and} \tag{46}$$

$$\text{Submodularity: } g_B(x) + g_I(x-1) - g_B(x-1) - g_I(x) \leq 0 \text{ for } 0 < x \leq m. \tag{47}$$

Threshold structure: For $g_y(x) \in \mathcal{C}_1$, we have

$$\text{Idle below a threshold: } g_I(x+1) + g_B(x-1) - g_B(x) - g_I(x) \geq 0 \text{ for } 0 < x < m. \tag{48}$$

For $g_y(x) \in \mathcal{C}_2 \cap \mathcal{C}_3$, we have

$$\text{Idle above a threshold: } g_I(x+1) + g_B(x-1) - g_B(x) - g_I(x) \leq 0 \text{ for } 0 < x < m. \tag{49}$$

Complementary property: For $g_y(x) \in \mathcal{C}_2$, we have

$$g_I(x+1) + g_B(x-2) - g_I(x) - g_B(x-1) \geq 0 \text{ for } 1 < x < m. \tag{50}$$

Some combinations of relations allow us to reduce the number of relations, which need to be proven. For $\mathcal{C}_1$, the sum of (48) and (44) leads to the convexity property of $V_k(x, I)$ (Relation (43)). For $\mathcal{C}_2$, the sum of (49) and (50) leads to the convexity property of $V_k(x, B)$ (Relation (42)); furthermore the sum of (49) and (50), where x is incremented by 1, leads to the convexity property of $V_k(x, I)$ (Relation (43)). Finally, the sum of (49) and (42) leads to (44). For $\mathcal{C}_3$, the concavity of $V_k(x, B)$ and $V_k(x, I)$ (Relations (45) and (46)) can be deduced from (47) and (49).

C.2 Induction step for the first order properties

First, we show the propagation of (40). For $0 \leq x < m$ and $y = I, B$, we have

$$
\begin{aligned}
V_{k+1}(x+1,y) - V_{k+1}(x,y) &= f_y(x+1) - f_y(x) + \lambda \mathbb{1}_{x<m-1,y=B}(V_k(x+2,B) - V_k(x+1,B)) \\
&+ \lambda \mathbb{1}_{x<m-1,y=I}(W_k(x+2) - W_k(x+1)) \\
&+ \lambda \mathbb{1}_{x=m-1,y=B}(V_k(m,B) - W_k(m-1)) \\
&+ \lambda \mathbb{1}_{x=m-1,y=I}(V_k(m,B) - W_k(m)) \\
&+ \mu \mathbb{1}_{y=B}(W_k(x+1) - W_k(x)) \\
&+ x\gamma \mathbb{1}_{y=B}(V_k(x,B) - V_k(x-1,B)) + \gamma \mathbb{1}_{y=B} V_k(x,B) \\
&+ x\gamma \mathbb{1}_{y=I}(W_k(x) - W_k(x-1)) + \gamma \mathbb{1}_{y=I} W_k(x) \\
&+ (1 - \lambda - \mu \mathbb{1}_{y=B} - (x+1)\gamma)(\mathbb{1}_{y=B}(V_k(x+1,B) - V_k(x,B)) + \mathbb{1}_{y=I}(W_k(x+1) - W_k(x))) \\
&- \gamma \mathbb{1}_{y=B} V_k(x,B) - \gamma \mathbb{1}_{y=I} W_k(x).
\end{aligned}
$$

The terms proportional with $\lambda \mathbb{1}_{x<m-1,y=B}$ (line 1), $x\gamma \mathbb{1}_{y=B}$ (line 6), and $\mathbb{1}_{y=B}(1 - \lambda - \mu \mathbb{1}_{y=B} - (x+1)\gamma)$ (line 8) are positive since (40) holds for $V_k(x,B)$. For $0 \leq x < m$, we either have $W_k(x+1) = V_k(x+1,I)$ and $W_k(x+1) - W_k(x) \geq V_k(x+1,I) - V_k(x,I) \geq 0$ or $W_k(x+1) = V_k(x,B)$ and $W_k(x+1) - W_k(x) \geq V_k(x,B) - V_k(x,I) \geq 0$ due to Relation (41). Therefore, $W_k(x+1) \geq W_k(x)$. This proves that the terms proportional with $\lambda \mathbb{1}_{x<m-1,y=I}$ (line 2), $\mu \mathbb{1}_{y=B}$ (line 5), $x\gamma \mathbb{1}_{y=I}$ (line 7) and $\mathbb{1}_{y=I}(1-\lambda-\mu \mathbb{1}_{y=B}-(x+1)\gamma)$ (line 8) are positive. The terms proportional with $\lambda \mathbb{1}_{x=m-1,y=B}$ (line 3) and $\lambda \mathbb{1}_{x=m-1,y=I}$ (line 4) are also positive since $V_k(m,B) - W_k(m-1) \geq V_k(m,B) - W_k(m) \geq V_k(m,B) - V_k(m-1,B) \geq 0$. Finally, the remaining terms proportional with γ (lines 6, 7 and 9) sum up to zero. This proves that (40) holds for V_{k+1}.

Next, we have for $0 \leq x \leq m$,

$$
\begin{aligned}
V_k k+1(x,B) - V_{k+1}(x,I) &= f_B(x) - f_I(x) + P + \lambda \mathbb{1}_{x<m}(V_k(x+1,B) - W_k(x+1)) \\
&+ \lambda \mathbb{1}_{x=m}(V_k(m,B) - W_k(m-1)) \\
&+ \mu W_k(x) + x\gamma(V_k(x-1,B) - W_k(x-1)) \\
&+ (1 - \mu - x\gamma)(V_k(x,B) - W_k(x)) - \mu W_k(x).
\end{aligned}
$$

The term proportional with $\lambda \mathbb{1}_{x<m}$ (line 1) is positive since $V_k(x+1,B) - W_k(x+1) \geq V_k(x+1,B) - V_k(x+$

$1, I) \geq 0$. With the same approach, we show that the terms proportional with $x\gamma$ and $1-\mu-x\gamma$ (lines 3 and 4) are also positive. The term proportional with$\lambda \mathbb{1}_{x=m}$ (line 2) is also positive since $V_k(m,B)-W_k(m-1) \geq V_k(m,B)-W_k(m)$. Finally, the remaining terms in μ sum up to zero. This proves that (41) holds for V_{k+1}.

C.3 Induction step for the second order and threshold properties

Second order properties if $\mu \geq \gamma$ and $f_y(x) \in \mathcal{C}_1$. Assume that (48) holds for V_k. We show that (48) holds for V_{k+1}. For $0 < x < m$, we have

$$
\begin{aligned}
V_{k+1}(x+1,I) + V_{k+1}(x-1,B) - V_{k+1}(x,B) - V_{k+1}(x,I) &= f_I(x+1) + f_B(x-1) - f_B(x) - f_I(x) \\
&+ \lambda \mathbb{1}_{x<m-1} \left(W_k(x+2) + V_k(x,B) - V_k(x+1,B) - W_k(x+1)\right) \\
&+ \lambda \mathbb{1}_{x=m-1} \left(V_k(m-1,B) - W_k(m)\right) \\
&+ \mu(W_k(x-1) - W_k(x)) \\
&+ (x-1)\gamma(W_k(x) + V_k(x-2,B) - V_k(x-1,B) - W_k(x-1)) \\
&+ \gamma(2W_k(x) - V_k(x-1,B) - W_k(x-1)) \\
&+ (1-\lambda-\mu-(x+1)\gamma)(W_k(x+1) + V_k(x-1,B) - V_k(x,B) - W_k(x)) \\
&+ \mu(W_k(x+1) - W_k(x)) \\
&+ \gamma(2V_k(x-1,B) - V_k(x,B) - W_k(x)).
\end{aligned}
$$

Consider the term proportional with $\lambda \mathbb{1}_{x<m-1}$ (line 2) on the right hand side of this equation. If $W_k(x+2) = V_k(x+2,I)$, then due to $V_k(x+1,B) + W_k(x+1) \leq V_k(x+1,B) + V_k(x+1,I)$, we have

$$
\begin{aligned}
&W_k(x+2) + V_k(x,B) - V_k(x+1,B) - W_k(x+1) \\
&\qquad \geq V_k(x+2,I) + V_k(x,B) - V_k(x+1,B) - V_k(x+1,I) \geq 0,
\end{aligned}
$$

since (48) holds for V_k. If $W_k(x+2) = V_k(x+1,B)$, then due to $V_k(x+1,B) + W_k(x+1) \leq V_k(x+1,B) + V_k(x,B)$, we have

$$
\begin{aligned}
&W_k(x+2) + V_k(x,B) - V_k(x+1,B) - W_k(x+1) \\
&\qquad \geq V_k(x+1,B) + V_k(x,B) - V_k(x+1,B) - V_k(x,B) = 0.
\end{aligned}
$$

This proves that the term proportional with $\lambda \mathbb{1}_{x<m-1}$ is positive. With the same approach, we prove that the term proportional with $(x-1)\gamma$ (line 5) and the one proportional with $1-\lambda-\mu-(x+1)\gamma$ (line 7) are also positive. The term proportional with $\lambda \mathbb{1}_{x=m-1}$ (line 3) is positive since $V_k(m-1,B) \geq W_k(m)$. The remaining terms (lines 4, 6, 8 and 9) can be rewritten as

$$
\begin{aligned}
&\gamma(W_k(x+1) + V_k(x-1,B) - V_k(x,B) - W_k(x)) \\
&\quad + (\mu-\gamma)(W_k(x+1) + W_k(x-1) - 2W_k(x)).
\end{aligned}
$$

The first term proportional with γ is positive as it is the same one as the one proportional with $1-\lambda-\mu-(x+1)\gamma$ (line 7) in the last equation. Consider now the term proportional with $\mu-\gamma$.

Case 1: $W_k(x+1) + W_k(x-1) = V_k(x,B) + V_k(x-2,B)$. Since $2W_k(x) \leq 2V_k(x-1,B)$, the convexity of $V_k(x,B)$ in x proves that the term proportional with $\mu-\gamma$ is positive.

Case 2: $W_k(x+1) + W_k(x-1) = V_k(x+1,I) + V_k(x-1,I)$. Since $2W_k(x) \leq 2V_k(x,I)$, the convexity of $V_k(x,I)$ proves that the term proportional with $\mu-\gamma$ is positive.

Case 3: $W_k(x+1) + W_k(x-1) = V_k(x,B) + V_k(x-1,I)$. Since $2W_k(x) \leq V_k(x,I) + V_k(x-1,B)$, Relation (44) proves that the term proportional with $\mu-\gamma$ is positive. Note that the case $W_k(x+1) + W_k(x-1) = V_k(x+1,I) + V_k(x-2,B)$ should not be considered as it is in contradiction with (48).

We now prove that (42) holds for V_{k+1}. For $0 < x < m$, we obtain

$$
\begin{aligned}
V_{k+1}(x+1,B) + V_{k+1}(x-1,B) - 2V_{k+1}(x,B) &= f_B(x+1) + f_B(x-1) - 2f_B(x) \\
&+ \lambda \mathbb{1}_{x<m-1}(V_k(x+2,B) + V_k(x,B) - 2V_k(x+1,B)) \\
&\quad + \lambda \mathbb{1}_{x=m-1}(-W_k(m-1) + V_k(m-1,B)) \\
&\quad + \mu(W_k(x+1) + W_k(x-1) - 2W_k(x)) \\
&\quad + (x-1)\gamma(V_k(x,B) + V_k(x-2,B) - 2V_k(x-1,B)) \\
&\quad + 2\gamma(V_k(x,B) - V_k(x-1,B)) \\
&\quad + (1-\lambda-\mu-(x+1)\gamma)(V_k(x+1,B) + V_k(x-1,B) - 2V_k(x,B)) \\
&\quad + 2\gamma(V_k(x-1,B) - V_k(x,B)).
\end{aligned}
$$

The terms proportional with $\lambda \mathbb{1}_{x<m-1}$ (line 2), $(x-1)\gamma$ (line 5), and $1-\lambda-\mu-(x+1)\gamma$ (line 7) are positive since (42) holds for V_k. The term proportional with μ (line 4) is also positive due to (48), (42) and (44) as shown for the induction from V_k to V_{k+1} for (48). The term proportional with $\lambda \mathbb{1}_{x=m-1}$ is also positive

since $V_k(x,B)$ is increasing in x. Specifically, $-W_k(m-1)+V_k(m-1,B) \geq V_k(m-1,B)-V_k(m-2,B) \geq 0$. Finally, the remaining terms proportional with 2γ (lines 6 and 8) sum up to zero. This proves the induction step from V_k to V_{k+1} for (42).

We consider now (44) and prove the induction step for this relation. For $0 < x \leq m$, we obtain

$$\begin{aligned}
V_{k+1}(x,B) + V_{k+1}(x-1,I) - V_{k+1}(x-1,B) - V_{k+1}(x,I) &= f_B(x) + f_I(x-1) - f_B(x-1) - f_I(x) \\
&+ \lambda \mathbb{1}_{x<m}(V_k(x+1,B) + W_k(x) - V_k(x,B) - W_k(x+1)) \\
&+ \mathbb{1}_{x=m}(-W_k(m-1) + W_k(m)) \\
&+ \mu(W_k(x) - W_k(x-1)) \\
&+ (x-1)\gamma(V_k(x-1,B) + W_k(x-2) - V_k(x-2,B) - W_k(x-1)) \\
&+ \gamma(V_k(x-1,B) - W_k(x-1)) \\
&+ (1-\lambda-\mu-x\gamma)(V_k(x,B) + W_k(x-1) - V_k(x-1,B) - W_k(x)) \\
&+ \gamma(W_k(x-1) - V_k(x-1,B)).
\end{aligned}$$

We first consider the term proportional with $\lambda \mathbb{1}_{x<m}$ (line 2).

Case 1: $W_k(x) = V_k(x,I)$. In this case, we have

$$V_k(x+1,B) + W_k(x) - V_k(x,B) - W_k(x+1) \geq V_k(x+1,B) + V_k(x,I) - V_k(x,B) - V_k(x+1,I) \geq 0,$$

since (44) holds for V_k.

Case 2: $W_k(x) = V_k(x-1,B)$ In this case, we have

$$V_k(x+1,B) + W_k(x) - V_k(x,B) - W_k(x+1) \geq V_k(x+1,B) + V_k(x-1,B) - 2V_k(x,B) \geq 0,$$

since (42) holds for V_k. This proves that the term proportional with $\lambda \mathbb{1}_{x<m}$ (line 2) is positive. With the same approach, we prove that the terms proportional with $(x-1)\gamma$ (line 5) and $1-\lambda-\mu-x\gamma$ (line 7) are also positive. The term proportional with $\lambda \mathbb{1}_{x=m}$ (line 3) is positive since $W_k(x)$ is increasing in x. Finally, the remaining terms proportional with γ (lines 6 and 8) sum up to zero. This proves that (44) holds for V_{k+1}.

Second order properties in the case $\gamma \geq \mu$, $\lambda \geq 2\gamma$ and $f_y(x) \in \mathcal{C}_2$. In this case, we need to redefine the artificial terms $V_k(m+1,B)$ and $V_k(m+1,I)$ after a $\lambda-$transition from state $x = m$ in order to satisfy the relations involved in this case. One solution is to modify the $\lambda-$terms for $x = m$ in (4) into the term $W_k(m) + V_k(m-1, B) - V_k(m-2, B)$ for a $\lambda-$transition from state (m, I) and into the term $V_k(m, B) + V_k(m-1, B) - V_k(m-2, B)$ for a $\lambda-$transition from state (m, B). We can check easily that the first order monotonicity properties (Relations (40) and (41)) hold with this change. Other alternatives are possible. We do not detail the transitions at the boundary state in what follows.

Consider now the propagation of (49) from V_k to V_{k+1}. For $0 < x < m$, we have

$$
\begin{aligned}
V_{k+1}(x,B) + V_{k+1}(x,I) - V_{k+1}(x+1,I) - V_{k+1}(x-1,B) &= f_B(x) + f_I(x) - f_I(x+1) - f_B(x-1) \\
&+ \lambda\left(V_k(x+1,B) + W_k(x+1) - W_k(x+2) - V_k(x,B)\right) \\
&+ \mu(W_k(x) - W_k(x-1)) \\
&+ (x-1)\gamma(V_k(x-1,B) - W_k(x-1) - W_k(x) - V_k(x-2,B)) \\
&+ \gamma(V_k(x-1,B) + W_k(x-1) - 2W_k(x)) \\
&+ (1-\lambda-\mu-(x+1)\gamma)(V_k(x,B) - W_k(x) - W_k(x+1) - V_k(x-1,B)) \\
&+ \mu(W_k(x) - W_k(x+1)) \\
&+ \gamma(V_k(x,B) + W_k(x) - 2V_k(x-1,B)).
\end{aligned}
$$

Consider the term proportional with λ (line 2) on the right hand side of this equation. If $W_k(x+1) = V_k(x+1, I)$, then due to $V_k(x,B) + W_k(x+2) \leq V_k(x,B) + V_k(x+2,I)$, we have

$$
\begin{aligned}
&W_k(x+1) + V_k(x+1,B) - V_k(x,B) - W_k(x+2) \\
&\qquad \geq V_k(x+1,I) + V_k(x+1,B) - V_k(x,B) - V_k(x+2,I) \geq 0,
\end{aligned}
$$

since (49) holds for V_k. If $W_k(x+1) = V_k(x,B)$, then due to $V_k(x,B)+W_k(x+2) \leq V_k(x,B)+V_k(x+1,B)$, we have

$$
\begin{aligned}
&W_k(x+1) + V_k(x+1,B) - V_k(x,B) - W_k(x+2) \\
&\qquad \geq V_k(x,B) + V_k(x+1,B) - V_k(x,B) - V_k(x+1,B) = 0.
\end{aligned}
$$

This proves that the term proportional with λ is positive. With the same approach, we prove that the term proportional with $(x-1)\gamma$ (line 4) and the one proportional with $1-\lambda-\mu-(x+1)\gamma$ (line 6) are also positive. The remaining terms (lines 3, 5, 7 and 8) can be rewritten as

$$\mu(W_k(x) + V_k(x, B) - W_k(x+1) - V_k(x-1, B))$$
$$+ (\gamma - \mu)(W_k(x-1) + V_k(x, B) - V_k(x-1, B) - W_k(x)).$$

The first term proportional with μ is positive as it is the same one as the one proportional with $1 - \lambda - \mu - (x+1)\gamma$ (line 6) in the last equation. Consider now the term proportional with $\gamma - \mu$.

Case 1: If $W_k(x-1) = V_k(x-1, I)$, then

$$W_k(x-1) + V_k(x, B) - V_k(x-1, B) - W_k(x) \geq V_k(x-1, I) + V_k(x, B) - V_k(x-1, B) - V_k(x, I) \geq 0,$$

since (44) holds for V_k.

Case 2: If $W_k(x-1) = V_k(x-2, B)$, then

$$W_k(x-1) + V_k(x, B) - V_k(x-1, B) - W_k(x) \geq V_k(x-2, B) + V_k(x, B) - 2V_k(x-1, B) \geq 0,$$

since (42) holds for V_k. This proves that (49) holds for V_{k+1} under the condition $\gamma \geq \mu$.

We now consider (50). For $1 < x < m$, we get

$$V_{k+1}(x+1, I) + V_{k+1}(x-2, B) - V_{k+1}(x, I) - V_{k+1}(x-1, B) = f_I(x+1) + f_B(x-2) - f_I(x) - f_B(x-1)$$
$$+ \lambda(W_k(x+2) + V_k(x-1, B) - W_k(x+1) - V_k(x, B))$$
$$+ \mu(W_k(x-2) - W_k(x-1))$$
$$+ (x-2)\gamma(W_k(x) + V_k(x-3, B) - W_k(x-1) - V_k(x-2, B))$$
$$+ \gamma(3W_k(x) - 2W_k(x-1) - V_k(x-2, B))$$
$$+ (1 - \lambda - \mu - (x+1)\gamma)(W_k(x+1) + V_k(x-2, B) - W_k(x) - V_k(x-1, B))$$
$$+ \mu(W_k(x+1) - W_k(x))$$
$$+ \gamma(3V_k(x-2, B) - W_k(x) - 2V_k(x-1, B))$$

Consider the term proportional with λ (line 2) on the right hand side of this equation. If $W_k(x+2) =$

$V_k(x+2,I)$, then due to $V_k(x,B)+W_k(x+1) \leq V_k(x,B)+V_k(x+1,I)$, we have

$$\begin{aligned} &W_k(x+2)+V_k(x-1,B)-V_k(x,B)-W_k(x+1) \\ &\qquad \geq V_k(x+2,I)+V_k(x-1,B)-V_k(x,B)-V_k(x+1,I) \geq 0, \end{aligned}$$

since (50) holds for V_k. If $W_k(x+2)=V_k(x+1,B)$, then due to $V_k(x,B)+W_k(x+1) \leq 2V_k(x,B)$, we have

$$W_k(x+2)+V_k(x-1,B)-V_k(x,B)-W_k(x+1) \geq V_k(x+1,B)+V_k(x-1,B)-2V_k(x,B) \geq 0,$$

since V_k is convex in x. This proves that the term proportional with $\lambda \mathbb{1}_{x<m-1}$ is positive. With the same approach, we prove that the term proportional with $(x-2)\gamma$ (line 4) and the one proportional with $1-\lambda-\mu-(x+1)\gamma$ (line 6) are also positive. By summing up the terms proportional with μ (lines 3 and 7), we obtain

$$W_k(x+1)+W_k(x-2)-W_k(x)-W_k(x-1).$$

To prove that this relation is positive, we need to prove that $W_k(x)$ is convex in x. This mans that we need to prove that

$$W_k(x+1)+W_k(x-1)-2W_k(x) \geq 0.$$

The cases $W_k(x+1)+W_k(x-1)=V_k(x+1,I)+V_k(x-1,I)$ and $W_k(x+1)+W_k(x-1)=V_k(x,B)+V_k(x-2,B)$ can be proven with the convexity property of $V_k(x,y)$ for $y=I,B$. The case $W_k(x+1)+W_k(x-1)=V_k(x+1,I)+V_k(x-2,B)$ can be proven using Relation (50). Finally, the case $W_k(x+1)+W_k(x-1)=V_k(x,B)+V_k(x-1,I)$ should not be considered as it is in contradiction with (49). This proves that $W_k(x+1)+W_k(x-2)-W_k(x)-W_k(x-1) \geq 0$. Consider now the sum of the λ term (line 2) and the remaining ones in γ (lines 5 and 8). We obtain

$$\begin{aligned} &(\lambda-2\gamma)(W_k(x+2)+V_k(x-1,B)-W_k(x+1)-V_k(x,B)) \\ &+2\gamma(W_k(x+2)+W_k(x)+V_k(x-2,B)-V_k(x,B)-W_k(x-1)-W_k(x+1). \end{aligned}$$

The term proportional with $\lambda-2\gamma$ is positive (it is the same one as the one proportional with λ in the last equation). We now prove that the term proportional with 2γ is positive.

Case 1: $W_k(x+2) + W_k(x) = V_k(x+2, I) + V_k(x, I)$. In this case, we have

$$\begin{aligned}
&W_k(x+2) + W_k(x) + V_k(x-2, B) - V_k(x, B) - W_k(x-1) - W_k(x+1) \\
&\quad \geq V_k(x+2, I) + V_k(x, I) + V_k(x-2, B) - V_k(x, B) - V_k(x-1, I) - V_k(x+1, I) \\
&\quad \geq V_k(x+1, I) + V_k(x-2, B) - V_k(x, B) - V_k(x-1, I) \quad \text{(Convexity of } V_k(x, I)), \\
&\quad \geq V_k(x+1, I) + V_k(x-2, B) - V_k(x-1, B) - V_k(x, I) \geq 0, \quad \text{(Relation (44))}
\end{aligned}$$

since (50) holds for V_k.

Case 2: $W_k(x+2) + W_k(x) = V_k(x+1, B) + V_k(x-1, B)$. In this case, we have

$$\begin{aligned}
&W_k(x+2) + W_k(x) + V_k(x-2, B) - V_k(x, B) - W_k(x-1) - W_k(x+1) \\
&\quad \geq V_k(x+1, B) + V_k(x-1, B) - 2V_k(x, B) \geq 0,
\end{aligned}$$

since V_k is convex in x.

Case 3: $W_k(x+2) + W_k(x) = V_k(x+2, I) + V_k(x-1, B)$. In this case, we have

$$\begin{aligned}
&W_k(x+2) + W_k(x) + V_k(x-2, B) - V_k(x, B) - W_k(x-1) - W_k(x+1) \\
&\quad \geq V_k(x+2, I) + V_k(x-1, B) - V_k(x, B) - V_k(x+1, I) \geq 0,
\end{aligned}$$

since (50) holds for V_k. The case $W_k(x+2) + W_k(x) = V_k(x+1, B) + V_k(x, I)$ should not be considered as it is in contradiction with (49).

Second order properties in the case $\mu \geq \gamma$ and $f_y(x) \in \mathcal{C}_3$. We omit the redefinition of the boundary transition for this case. We first consider the propagation of (49) from V_k to V_{k+1}. For $0 < x < m$, we have

$$
\begin{aligned}
V_{k+1}(x,B) + V_{k+1}(x,I) - V_{k+1}(x+1,I) - V_{k+1}(x-1,B) &= f_B(x) + f_I(x) - f_I(x+1) - f_B(x-1) \\
&\quad + \lambda\left(V_k(x+1,B) + W_k(x+1) - W_k(x+2) - V_k(x,B)\right) \\
&\quad + \mu(W_k(x) - W_k(x-1)) \\
&\quad + (x-1)\gamma(V_k(x-1,B) - W_k(x-1) - W_k(x) - V_k(x-2,B)) \\
&\quad + \gamma(V_k(x-1,B) + W_k(x-1) - 2W_k(x)) \\
&\quad + (1-\lambda-\mu-(x+1)\gamma)(V_k(x,B) - W_k(x) - W_k(x+1) - V_k(x-1,B)) \\
&\quad + \mu(W_k(x) - W_k(x+1)) \\
&\quad + \gamma(V_k(x,B) + W_k(x) - 2V_k(x-1,B)).
\end{aligned}
$$

With the same approach as in the case where $f_y(x) \in \mathcal{C}_2$, we prove that the term proportional with λ (line 2), $(x-1)\gamma$ (line 4) and $1-\lambda-\mu-(x+1)\gamma$ (line 6) are positive. The remaining terms (lines 3, 5, 7 and 8) can be rewritten as

$$
\begin{aligned}
&(\mu-\gamma)(2W_k(x) - W_k(x+1) - W_k(x-1)) \\
&\qquad + \gamma(V_k(x,B) + W_k(x) - W_k(x+1) - V_k(x-1,B)).
\end{aligned}
$$

Consider the term proportional with $\mu - \gamma$.

Case 1: $W_k(x) = V_k(x,I)$. Since $W_k(x+1) + W_k(x-1) \leq V_k(x+1,I) + V_k(x-1,I)$ and $V_k(x,I)$ is concave in x (Relation (46)), then the term proportional with $\mu-\gamma$ is positive.

Case 2: $W_k(x) = V_k(x-1,B)$. Since $W_k(x+1) + W_k(x-1) \leq V_k(x,B) + V_k(x-2,B)$ and $V_k(x,B)$ is concave in x (Relation (45)), then the term proportional with $\mu-\gamma$ is positive. The term proportional with γ is also positive for the same reason as those in λ, $(x-1)\gamma$ and $1-\lambda-\mu-(x+1)\gamma$ in the first equation.

We now consider Relation (47). For $0 < x < m$, we have

$$
\begin{aligned}
V_{k+1}(x,I) + V_{k+1}(x-1,B) - V_{k+1}(x,B) - V_{k+1}(x-1,I) &= f_I(x) + f_B(x-1) - f_B(x) - f_I(x-1) \\
&+ \lambda\left(W_k(x+1) + V_k(x,B) - V_k(x+1,B) - W_k(x)\right) \\
&+ \mu(W_k(x-1) - W_k(x)) \\
&+ (x-1)\gamma(W_k(x-1) + V_k(x-2,B) - V_k(x-1,B) - W_k(x-2)) \\
&+ \gamma(W_k(x-1) - V_k(x-1,B)) \\
&+ (1-\lambda-\mu-x\gamma)(W_k(x) + V_k(x-1,B) - V_k(x,B) - W_k(x-1)) \\
&+ \mu(W_k(x) - W_k(x-1)) \\
&+ \gamma(V_k(x-1,B) - W_k(x-1)).
\end{aligned}
$$

The right hand side of this equation is positive. Specifically, lines 2, 4, and 6 are positive since (45) and (47) hold for V_k, Finally, the sum of lines 3, 5, 7 and 8 is zero. This finishes the proof of the theorem. □

D Proof of Lemma 2

Proof. Let us start with Equation (21). We need to determine two independent solutions for this equation. For this purpose, we introduce a function $\mathcal{F}(z)$ for $z \in \mathbb{C}$ and we express w_x as

$$w_x = \int_\zeta z^{-(x+1)} \mathcal{F}(z) \mathrm{d}z,$$

where ζ is a contour such that there are no boundary contributions arising in the integral from endpoints of ζ. This allows us to use the integration by part and show that $x w_x = \int_\zeta z^{-(x+1)} z \mathcal{F}'(z) \mathrm{d}z$. Equation (21) can then be rewritten as

$$\int_\zeta z^{-(x+1)} \left(\mathcal{F}(z)\left[(a(1-z)+s)\right] + \mathcal{F}'(z)(z-1)\right) \mathrm{d}z = 0.$$

Therefore, $\mathcal{F}(z)$ is one solution of the differential equation

$$\mathcal{F}(z)\left[(a(1-z)+s)\right] + \mathcal{F}'(z)(z-1) = 0.$$

Consequently, $\mathcal{F}(z)$ is proportional with $e^{az}(1-z)^{-s}$. We thus determine two independent solutions of (21) by selecting two different contours encircling $z=0$. We consider the contour ζ_1 defined as a small circle in the z-plane, on which $|z|<1$, and ζ_2 which goes from $-\infty - i\epsilon$ to $-\infty + i\epsilon$ for $\epsilon > 0$, encircling $z=1$ in the counterclockwise sense. This provides two independent solutions of (21), A_x and B_x, as expressed in (24) and (25). Note that for (24) the integrand is analytic inside the unit circle, as we consider $(1-z)^s = |1-z|^s e^{is\arg(1-z)}$, with $|\arg(1-z)|<\pi$, such that for $z \in \mathbb{R}$ and $z<1$, $\arg(1-z)=0$. For (25), we use the branch $(z-1)^s = |z-1|^s e^{is\arg(z-1)}$, where $|\arg(z-1)|<\pi$, so the integrand is analytic in $\mathbb{C} - \{\mathrm{Im}(z)=0, \mathrm{Re}(z)<1\}$.

The expression of A_x in (24) can be obtained explicitly. By expanding $(1-z)^{-s}$, we obtain $(1-z)^{-s} = 1 + sz + s(s+1)\frac{z^2}{2!} + \cdots$. Therefore, we deduce that

$$A_x = \sum_{k=0}^{x} \frac{a^{x-k}}{k!(x-k)!} \frac{\Gamma(s+k)}{\Gamma(s)}.$$

We now consider Equation (22). One known solution of (22) from the analysis of the M/M/s queue is $C_x = \frac{a^x}{x!}$. We check that a second and independent solution of (22) is $D_x = \sum_{k=0}^{x-1} \frac{a^{x-k}k!}{x!}$. Finally we consider Equation (23) and check that E_x and F_x are two independent solutions of this equation. □

E Proof of Lemma 3

Proof. We consider the asymptotic expressions of A_x and B_x as x tends to infinity. For A_x, we set $z = 1 - \frac{y}{x}$ as A_x is governed by the singularity at $z=1$. We thus deduce that

$$A_x \underset{x\to\infty}{\sim} \frac{1}{2\pi i} \int_{\overline{\zeta_1}} e^a e^y y^{-s} x^{s-1} \mathrm{d}\, y = \frac{e^a x^{s-1}}{\Gamma(s)},$$

where $\overline{\zeta_1}$ is the change of the contour ζ_1 through the change of variable $z = 1 - \frac{y}{x}$. For B_x, we dilate the contour ζ_2 such that $|z| >> 1$ and then use $|z-1| \sim |z|$. This leads to

$$B_x \underset{x\to\infty}{\sim} \frac{1}{2i\pi} \int_{\zeta_2} z^{-(x+1+s)} e^{az} \mathrm{d}z = \frac{a^{x+s}}{\Gamma(x+1+s)}.$$

We next write

$$\frac{A_m}{B_m} \underset{m\to\infty}{\sim} \frac{e^a m^{s-1}\Gamma(m+1+s)}{\Gamma(s)a^{m+s}} \underset{m\to\infty}{\sim} \frac{e^a m^{2s-1} m!}{a^{m+s}\Gamma(s)} \underset{m\to\infty}{\sim} \frac{\sqrt{2\pi}e^a(m+s)^{m+s}}{\Gamma(s)}.$$

This proves that $\lim_{m\to\infty} \frac{A_m}{B_m} = \infty$. □

F Proof of Theorem 2

Proof. **Policy π_0.** Let us start with Policy π_0. We observe that Equations (7) and (11) correspond to (23) and Equation (9) corresponds to (22).

We first consider (9). From Theorem 2, we can express p_x^0 as $p_x^0 = c_1 C_x + c_2 D_x$ for $0 \leq x \leq n$ where the constants c_1 and c_2 need to be determined. Using the boundary equations for $x = 0$ and $x = 1$, we obtain

$$p_0^0 = c_1, \text{ and } p_1^0 = c_1 a + c_2 a.$$

We then express p_x^0 in p_0^0 and p_1^1 as

$$p_x^0 = p_0^0 C_x + \frac{p_1^0 - a p_0^0}{a} D_x \text{ for } 0 \leq x \leq n.$$

We now relate p_x^0 with q_0^0. Using (8) and $s q_0^0 = a p_n^0$, we obtain

$$p_0^0 = q_0^0 \frac{s}{a} \frac{1 + D_n}{C_n}, \text{ and } p_1^0 = q_0^0 s \frac{1 + D_n - C_n}{C_n}.$$

We thus deduce that

$$p_x^0 = q_0^0 \frac{s}{a} \left(\frac{C_x(1 + D_n)}{C_n} - D_x \right) \text{ for } 0 \leq x \leq n.$$

Consider now (7). From Theorem 2, we can express q_x^0 as

$$q_x^0 = c_3 E_x + c_4 F_x \text{ for } 0 \leq x \leq n.$$

Using the boundary conditions at $x = 0$ and $x = 1$, we deduce that

$$\begin{aligned}
(s + a) q_0^0 &= (s + 1) q_1^0, \\
q_0^0 &= c_3 \frac{a^s}{\Gamma(s + 1)} + c_4, \text{ and} \\
q_1^0 &= c_3 \frac{a^{s+1}}{\Gamma(s + 2)} + c_4 \frac{s + a}{s + 1}.
\end{aligned}$$

By solving this system, we find that $c_3 = 0$ and $q_x^0 = q_0^0 F_x$ for $0 \leq x \leq n$.

We observe that (11) can be simplified into

$$aq_x^0 = (x + 1 + s)q_{x+1}^0, \tag{51}$$

for $x \geq n$. This equation has only one solution that is proportional to $E_x = \frac{a^{x+s}}{\Gamma(x+1+s)}$ (i.e., one of the solutions of (23). Therefore, we deduce that $q_x^0 = \frac{E_x}{E_n} q_n^0 = \frac{E_x}{E_n} q_0^0 F_n$ for $x \geq n$.

The stationary probabilities are all expressed as functions of q_0^0. Using the normalizing condition, we deduce the expression of q_0^0.

Policy π_1. We now consider Policy π_1. We observe that Equations (12) corresponds to (21), (13) corresponds to (22), and (15) corresponds to (23).

We first consider (12). After expressing q_x^1 as a linear combination of A_x and B_x in (12), we find that the term in B_x is zero since $A_{-1} = 0$, $A_0 = 1$, and $A_1 = a + s$. Therefore, we have

$$q_x^1 = q_0^1 A_x \text{ for } 0 \leq x \leq n.$$

We now consider (13). Since $q_x^1 = q_0^1 A_x$ for $0 \leq x \leq n$, we write $p_x^1 = c_5 A_x + c_6 C_x + c_7 D_x$. By replacing the expression of p_x^1 in (13), we obtain

$$c_5 = q_0^1 \frac{sA_x}{(a + x)A_x - aA_{x-1} - (x + 1)A_{x+1}} = -q_0^1.$$

Using the boundary equation for $x = 0$, we deduce that $c_6 = p_0^1 + q_0^1$. Also, since $p_{n+1}^1 = 0$, we get $c_7 = \frac{q_0^1 A_{n+1} - (p_0^1 + q_0^1) C_{n+1}}{D_{n+1}}$. Further, using $s \sum_{k=0}^{n} q_k^1 = ap_n^1$, we deduce that

$$q_0^1 \left(\frac{s}{a} \sum_{k=0}^{n} A_k D_{n+1} + A_n D_{n+1} - A_{n+1} D_n - C_n D_{n+1} + C_{n+1} D_n \right) = p_0^1 (D_{n+1} C_n - C_{n+1} D_n).$$

We have

$$C_n D_{n+1} - C_{n+1} D_n = \frac{a^n}{n!} \sum_{k=0}^{n} \frac{a^{n+1-k} k!}{(n+1)!} - \frac{a^{n+1}}{(n+1)!} \sum_{k=0}^{n-1} \frac{a^{n-k} k!}{n!} = \frac{a^{n+1}}{(n+1)!} = C_{n+1}.$$

Moreover, by summing up the equations $(s + a)A_0 = A_1$, $(s + a + 1)A_1 = aA_0 + 2A_2$, ..., $(s + a + n)A_n =$

$aA_{n-1} = (n+1)A_{n+1}$, we find that $s\sum_{k=0}^{n} A_k + aA_n = (n+1)A_{n+1}$. Therefore,

$$\frac{s}{a}\sum_{k=0}^{n} A_k D_{n+1} + A_n D_{n+1} = \frac{n+1}{a} D_{n+1} A_{n+1}.$$

Moreover, $1 + D_n = \frac{n+1}{a} D_{n+1}$. Thus, we obtain

$$p_0^1 = q_0^1 \frac{A_{n+1} - C_{n+1}}{C_{n+1}}.$$

From this expression, we express p_x^1 as

$$p_x^1 = q_0^1 \left(-A_x + \frac{A_{n+1}}{C_{n+1}} C_x \right) \text{ for } 0 \leq x \leq n.$$

Finally, (15) is identical to (11), thus we obtain $q_x^1 = \frac{E_x}{E_n} q_n^1 = \frac{E_x}{E_n} q_0^1 A_x$ for $x \geq n$.

The stationary probabilities are all expressed as functions of q_0^1. Using the normalizing condition, we deduce the expression of q_0^1.

Policy π_2. Equation (17) corresponds to (23), Equation (19) corresponds to (21), and Equation (20) corresponds to (22).

For (17), we express q_x^2 as a function of p_0^2 for $0 \leq x \leq n-2$. We obtain $q_x^2 = p_0^2 \frac{E_x}{E_{-1}}$ for $0 \leq x \leq n-2$.

For (19), we express q_x^2 as a linear combination of A_x and B_x:

$$q_x^2 = c_8 A_x + c_9 B_x \text{ for } x \geq n-1.$$

This leads to

$$c_8 = \frac{q_{n-1}^2 B_n - q_n^2 B_{n-1}}{B_n A_{n-1} - A_n B_{n-1}} \text{ and } c_9 = \frac{q_{n-1}^2 A_n - q_n^2 A_{n-1}}{-B_n A_{n-1} + A_n B_{n-1}}.$$

Using (30), we deduce that

$$c_8 = -\left(q_{n-1}^2 B_n - q_n^2 B_{n-1}\right) \frac{n!\Gamma(s)}{e^a a^{n+s-1}} \text{ and } c_9 = \left(q_{n-1}^2 A_n - q_n^2 A_{n-1}\right) \frac{n!\Gamma(s)}{e^a a^{n+s-1}}.$$

Assume that we truncate the system at state $m-1$ for a large value of m such that $\lim_{m\to\infty} q_m^2 = 0$.

Therefore, we get

$$\lim_{m\to\infty}\left(A_m\frac{q_{n-1}^2B_n-q_nB_{n-1}}{-U_n}+B_m\frac{q_{n-1}^2A_n-q_nA_{n-1}}{U_n}\right)=0,$$

which leads to

$$q_n^2=q_{n-1}^2\lim_{m\to\infty}\left(\frac{A_nB_m-B_nA_m}{A_{n-1}B_m-B_{n-1}A_n}\right).$$

From Lemma 3, we have $\lim_{m\to\infty}\frac{A_m}{B_m}=\infty$. Therefore, we have $q_n^2=q_{n-1}^2\frac{B_n}{B_{n-1}}$, which proves that $c_8=0$. From this analysis, we obtain that

$$q_x^2=q_{n-1}^2\frac{B_x}{B_{n-1}}\text{ for }x\geq n-1.$$

We now consider (20). Observing now that $a(p_x^2+q_x^2)=(x+1)(p_{x+1}^2+q_{x+1}^2)$ for $x\geq n$, we deduce that $p_x^2+q_x^2=(p_n^2+q_n^2)\frac{C_x}{C_n}$ for $x\geq n$. Furthermore, we have $aq_{n-1}^2=n(q_n^2+p_n^2)$. This leads to $p_x^2+q_x^2=q_{n-1}^2\frac{C_x}{C_{n-1}}$ for $x\geq n$. Using the expression of q_x^2 for $x\geq n-1$, we deduce that

$$p_x^2=q_{n-1}^2\left(\frac{C_x}{C_{n-1}}-\frac{B_x}{B_{n-1}}\right)\text{ for }x\geq n.$$

There remains to relate q_{n-1}^2 and p_0^2. Using (17) for $x=n-1$, we deduce that

$$ap_0^2\frac{E_{n-2}}{E_{-1}}+nq_{n-1}^2\frac{B_n}{B_{n-1}}=(a+s+n-1)q_{n-1}^2.$$

This leads to

$$q_{n-1}^2=p_0^2\frac{a\frac{E_{n-2}}{E_{-1}}}{a+s+n-1-n\frac{B_n}{B_{n-1}}}.$$

The stationary probabilities are then all expressed as functions of p_0^2. Using the normalizing condition, we deduce the expression of p_0^2. □

G Proof of Proposition 1

Proof. We prove (31) by induction on x. Note that we showed in Theorem 2 that A_x is a solution of the equation

$$(a+s+x)q_x = aq_{x-1} + (x+1)q_{x+1}.$$

Therefore, the term A_xB_0 is part of the expression of B_x. Relation (31) is valid for $x=-1$ as $A_0=1$. For $x=0$, we have $B_1=(s+a)B_0-aB_{-1}$, so Relation (31) is valid for $x=0$, with $h_{0,0}(s)=1$.

We now assume that (31) is valid for B_x and B_{x+1}. We deduce that

$$\begin{aligned}
B_{x+2} &= \frac{a+s+x+1}{x+2}B_{x+1} - \frac{a}{x+2}B_x \\
&= A_{x+2}B_0 - \frac{aB_{-1}}{(x+2)!}\left(\sum_{k=0}^{x} z^k h_{k,x}(s+x+1) + \sum_{k=1}^{x+1} z^k h_{k-1,x} - (x+1)\sum_{k=1}^{x} z^k h_{k-1,x-1}\right) \\
&= A_{x+2}B_0 \\
&- \frac{aB_{-1}}{(x+2)!}\left((s+x+1)h_{0,x} + \sum_{k=1}^{x} z^k\left(h_{k,x}(s+x+1) + h_{k-1,x} - (x+1)h_{k-1,x-1}\right) + z^{x+1}h_{x,x}\right).
\end{aligned}$$

We then define $h_{k,x+1}(s)$ by $h_{0,x+1}(s)=h_{0,x}(s+x+1)$, $h_{k,x+1}(s)=h_{k,x}(s+x+1)+h_{k-1,x}-(x+1)h_{k-1,x-1}$ for $1\le k\le x$, and $h_{x+1,x+1}(s)=h_{x,x}$ which proves the induction step as the relation $h_{0,x+1}(s)=h_{0,x}(s+x+1)$ with $h_{0,0}(s)=1$ leads to $h_{0,x+1}(s)=\frac{\Gamma(s+x+2}{\Gamma(s)}$ and $h_{x+1,x+1}(s)=h_{x,x}$ with $h_{0,0}(s)=1$ leads to $h_{x+1,x+1}(s)=1$. From the relation on $h_{k,x}(s)$, we deduce Relation (32). $\square$

H Proof of Proposition 2

Proof. Consider the equation

$$(a+s+x)q_x = aq_{x-1} + (x+1)q_{x+1}.$$

As explained in the proof of Theorem 2, the solutions of this equation are linear combinations of A_x and B_x. Instead of B_x, consider another solution of this equation, $\overline{B_x}$, such that $\overline{B_{-1}}=1$ and $\overline{B_0}=0$. With these values A_x and $\overline{B_x}$ are independent. Therefore, we have $B_x=\alpha A_x+\beta\overline{B_x}$. Consider now the Wronskian of A_x and $\overline{B_x}$ defined as $\overline{U_x}=A_x\overline{B_{x-1}}-A_{x-1}\overline{B_x}$ for $x\ge 0$. As in the proof of Theorem 2, we can show that

$\overline{U_x} = \overline{U_0}\frac{a^x}{x!}$ for $x \geq 0$. Since $\overline{U_0} = 1$, we deduce that $A_x\overline{B_{x-1}} - A_{x-1}\overline{B_x} = \frac{a^x}{x!}$. We then deduce that

$$\overline{B_x} = \frac{A_x\overline{B_{x-1}} - \frac{a^x}{x!}}{A_{x-1}} \text{ for } x \geq 1.$$

We deduce by induction from this expression that

$$\overline{B_x} = -A_x\sum_{k=1}^{x}\frac{a^k}{k!A_kA_{k-1}} \text{ for } x \geq 0.$$

For $x = 0$, this relation is valid as $\overline{B_0} = 0$. For $x = 1$, we have $\overline{B_1} = \frac{A_1\overline{B_0} - \frac{a^1}{1!}}{A_0} = -a$. Thus, the relation is valid. Assume now that the expression of $\overline{B_x}$ is valid. We prove that the same relation holds for $\overline{B_{x+1}}$. We have

$$\begin{aligned}\overline{B_{x+1}} &= \frac{A_{x+1}\overline{B_x} - \frac{a^{x+1}}{(x+1)!}}{A_x} = -A_{x+1}\sum_{k=1}^{x}\frac{a^k}{k!A_kA_{k-1}} - \frac{a^{x+1}}{A_x(x+1)!} \\ &= -A_{x+1}\sum_{k=1}^{x+1}\frac{a^k}{k!A_kA_{k-1}},\end{aligned}$$

which shows the induction step. In the expression of B_x, the coefficient β is found by replacing x by -1. We thus obtain $\beta = B_{-1} = \frac{e^a a^{s-1}}{\Gamma(s)}$. The coefficient α can be expressed as

$$\alpha = \frac{B_x - \beta\overline{B_x}}{A_x} \text{ for } x \geq 0.$$

We have $\frac{B_x}{A_x} \sim \frac{\Gamma(s)}{\sqrt{2\pi}e^a(s+x)^{s+x}}$ as x grows large. Therefore, $\lim_{x\to\infty}\frac{B_x}{A_x} = 0$. We also have $\frac{\overline{B_x}}{A_x} = -\sum_{k=1}^{x}\frac{a^k}{k!A_kA_{k-1}}$. For a large value of k we have $A_k \sim \frac{e^a k^{s-1}}{\Gamma(s)}$. As the sum $\sum_{k=1}^{m}\frac{(\Gamma(s))^2a^k}{k!e^{2a}k^{s-1}(k-1)^{s-1}}$ converges as m tends to infinity, the sum $\sum_{k=1}^{x}\frac{a^k}{k!A_kA_{k-1}}$ also converges as x tends to infinity. Thus, we obtain

$$\alpha = \frac{e^a a^{s-1}}{\Gamma(s)}\sum_{k=1}^{\infty}\frac{a^k}{k!A_kA_{k-1}}.$$

This leads to

$$B_x = \frac{e^a a^{s-1}}{\Gamma(s)}\left(A_x\sum_{k=1}^{\infty}\frac{a^k}{k!A_kA_{k-1}} - A_x\sum_{k=1}^{x}\frac{a^k}{k!A_kA_{k-1}}\right) = \frac{e^a a^{s-1}}{\Gamma(s)}A_x\sum_{k=x+1}^{\infty}\frac{a^k}{k!A_kA_{k-1}},$$

which allows us to express B_x explicitly. □

I Proof of Proposition 3

Proof. We want to determine two independent solutions for (34). To this end, we introduce $\Delta_x = T_x - T_{x-1}$. We rewrite Equation (34) by replacing x by $x+1$. We obtain

$$T_{x+1}(a + (x+1) + \theta) = aT_{x+2} + (x+1)T_x \text{ for } -1 \leq x \leq n-1. \tag{52}$$

The difference between (52) and (34) leads to

$$(a + x + 1 + \theta)\Delta_{x+1} = a\Delta_{x+2} + x\Delta_x \text{ for } 0 \leq x \leq n-1. \tag{53}$$

The solutions of (53) can be expressed in terms of confluent hypergeometric functions. We express one solution of (53) as $\Delta_x = \int_{z=0}^{1} G(z) z^{-(x+1)} \mathrm{d}z$, where $\lim_{z\to 0} G(z) z^{-x} = \lim_{z\to 1} G(z) z^{-x} = 0$. Using these boundary conditions and an integration by parts, we deduce that $x\Delta_x = \int_{z=0}^{1} G'(z) z z^{-(x+1)} \mathrm{d}z$. Therefore, (53) can be rewritten as

$$\int_{z=0}^{1} z^{-(x+1)} \left[\left((a+\theta) z^{-1} - a z^{-2} \right) G(z) + (1-z)\, G'(z) \right] \mathrm{d}z = 0.$$

Therefore, $G(z)$ is solution of

$$\frac{G'(z)}{G(z)} = \frac{a}{z^2} - \frac{\theta}{z} - \frac{\theta}{1-z}.$$

Thus, $G(z)$ is proportional with

$$G(z) = e^{-a/z} z^{-\theta} (1-z)^{\theta}.$$

Hence, one solution of (53) is

$$\Delta_x^1 = \int_{z=0}^{1} e^{-a/z} (1-z)^{\theta} z^{-(x+1+\theta)} \mathrm{d}z.$$

By changing the variable $y = \frac{1}{z}$, we get

$$\Delta_x^1 = \int_{y=1}^{\infty} e^{-ay} (y-1)^{\theta} y^{x-1} \mathrm{d}y.$$

Finally, we change the variable y by $u = y - 1$. We deduce that

$$\Delta_x^1 = e^a \int_{u=0}^{\infty} e^{-au} u^\theta (u+1)^{x-1} \mathrm{d}u = e^a \Gamma(a) M_2(\theta+1, x+\theta+1, a),$$

where M_2 is the Tricomi confluent hypergeometric function (i.e., the confluent hypergeometric function of the second kind).

A second solution of (53) can be found by considering the interval $(1, \infty)$ instead of $(0, 1)$ and replacing $1 - z$ by $z - 1$. For $x \geq 2$, we obtain

$$\begin{aligned}\Delta_x^2 &= \int_{z=1}^{\infty} e^{-a/z} (z-1)^\theta z^{-(x+1+\theta)} \mathrm{d}z = \int_{y=0}^{1} e^{-ay} (1-y)^\theta y^{x-1} \mathrm{d}y \\ &= \frac{\Gamma(x)\Gamma(1+\theta)}{\Gamma(x+1+\theta)} M_1(x, x+1+\theta, -a) = \frac{\Gamma(x)\Gamma(1+\theta)}{\Gamma(x+1+\theta)} e^{-a} M_1(1+\theta, x+1+\theta, a),\end{aligned}$$

where M_1 is the confluent hypergeometric function of the first kind. It should be noted that this second solution is valid only for $x \geq 2$. We now use $\theta T_x = a\Delta_x - x\Delta_{x+1}$. For Policies π_0 and π_1, since only Δ_x^1 satisfies (53) for $x = 0$ and $T_{n+1} = 1$, we deduce that

$$T_x = \frac{M_2(\theta, x+\theta+1, a)}{M_2(\theta, n+\theta+2, a)} = \frac{\sum_{k=0}^{x} \binom{x}{k} \frac{\Gamma(k+\theta)}{\Gamma(\theta) a^k}}{\sum_{k=0}^{n+1} \binom{n+1}{k} \frac{\Gamma(k+\theta)}{\Gamma(\theta) a^k}} \text{ for } 0 \leq x \leq n+1.$$

For Policy π_2, since Δ_x^1 tends to infinity as x tends to infinity, using $T_{n-1} = 1$, we deduce that

$$T_x = \frac{x!}{(n-1)!} \frac{\Gamma(n+\theta)}{\Gamma(x+1+\theta)} \frac{M_1(\theta, x+\theta+1, a)}{M_1(\theta, n+\theta, a)} = \frac{\int_0^1 e^{au} u^{\theta-1} (1-u)^x \mathrm{d}u}{\int_0^1 e^{au} u^{\theta-1} (1-u)^{n-1} \mathrm{d}u} \text{ for } x \geq n-1.$$

This finishes the proof of the proposition. □

J Proof of Proposition 4

Proof. **Policy π_1.** We start with Policy π_1. From Theorem 2, we express p_B^1 as

$$p_B^1 = \frac{\sum_{x=0}^{n} A_x + \frac{A_n}{E_n} \sum_{x=n+1}^{\infty} E_x}{\frac{A_{n+1}}{C_{n+1}} \sum_{x=0}^{n} C_x + \frac{A_n}{E_n} \sum_{x=n+1}^{\infty} E_x}.$$

From the expression of A_x as a sum in Lemma 2, we deduce that $A_x \underset{s\to\infty}{\sim} \frac{s^x}{x!}$. Therefore, $\sum_{x=0}^{n} A_x \underset{s\to\infty}{\sim} A_n \underset{s\to\infty}{\sim} \frac{s^n}{n!}$. For $x > n$, we have $\frac{E_x}{E_n} = \frac{a^{x-n}\Gamma(n+1+s)}{\Gamma(x+1+s)} \underset{s\to\infty}{\sim} \left(\frac{a}{s}\right)^{x-n}$. Therefore, we deduce that $\frac{A_n}{E_n}\sum_{x=n+1}^{\infty} E_x \underset{s\to\infty}{\sim} \frac{s^n}{n!}\frac{a/s}{1-a/s} \underset{s\to\infty}{\sim} \frac{as^{n-1}}{n!}$. This proves that the numerator of p_B^1 is equivalent to $\frac{s^n}{n!}$. At the denominator of p_B^1, the quantities C_x do not depend on s. Therefore, $\frac{A_{n+1}}{C_{n+1}}\sum_{x=0}^{n} C_x \underset{s\to\infty}{\sim} \frac{s^{n+1}}{(n+1)!}\frac{(n+1)!}{a^{n+1}}\sum_{k=0}^{n}\frac{a^x}{x!}$. The denominator of p_B^1 is then equivalent to $\left(\frac{s}{a}\right)^{n+1}\sum_{k=0}^{n}\frac{a^x}{x!}$. This proves the asymptotic expression of p_B^1 in Proposition 4.

Policy π_2. From Theorem 2, we have

$$p_B^2 = \frac{\sum_{x=0}^{n-2}\frac{E_x}{E_{-1}} + \frac{a\frac{E_{n-2}}{E_{-1}}}{a+s+n-1-n\frac{B_n}{B_{n-1}}}\sum_{x=n-1}^{\infty}\frac{B_x}{B_{n-1}}}{\sum_{x=-1}^{n-2}\frac{E_x}{E_{-1}} + \frac{a\frac{E_{n-2}}{E_{-1}}}{a+s+n-1-n\frac{B_n}{B_{n-1}}}\sum_{x=n-1}^{\infty}\frac{C_x}{C_{n-1}}}.$$

Using the expression of the Wronskian of A_x and B_x, and the asymptotic expression of A_x, we obtain $B_x \underset{s\to\infty}{\sim} \frac{s}{x}B_{x-1} - \frac{e^a a^s}{\Gamma(s)x}\left(\frac{a}{s}\right)^{x-1} \underset{s\to\infty}{\sim} \frac{s}{x}B_{x-1}$. Therefore, we deduce that $a+s+n-1-n\frac{B_n}{B_{n-1}} \underset{s\to\infty}{\sim} a+n-1$. Since $\sum_{x=n-1}^{\infty} z^{-(x+1)} = \frac{z^{-(n-1)}}{z-1}$, we have $\sum_{x=n-1}^{\infty} B_x = B_{n-2,s+1}$, where $B_{n-2,s+1}$ is equal to B_{n-2} by replacing s by $s+1$. Since $B_{n-2,s+1} \underset{s\to\infty}{\sim} B_{n-2}$, we have $\sum_{x=n-1}^{\infty}\frac{B_x}{B_{n-1}} \underset{s\to\infty}{\sim} \frac{B_{n-2}}{B_{n-1}} \underset{s\to\infty}{\sim} \frac{s}{a}$. We also have $\frac{E_x}{E_{-1}} = \frac{a^{x+1}\Gamma(s)}{\Gamma(x+1+s)} \underset{s\to\infty}{\sim} \left(\frac{a}{s}\right)^{x+1}$. Therefore, the numerator of p_B^2 is equivalent to $\frac{a}{s} + \left(\frac{a}{s}\right)^2 + \frac{a\left(\frac{a}{s}\right)^{n-2}}{a+n-1}$. Since C_x does not depend on x, the denominator is equivalent with $1 + \frac{a}{s} + \frac{a\frac{(n-1)!}{s^{n-1}}\sum_{x=n-1}^{\infty}\frac{a^x}{x!}}{a+n-1}$. This proves the asymptotic expression of p_B^2 in Proposition 4. $\square$

K Proof of Proposition 5

Proof. **Policy π_1.** We have

$$1 - p_B^1 = \frac{\frac{A_{n+1}}{C_{n+1}}\sum_{x=0}^{n} C_x - \sum_{x=0}^{n} A_x}{\frac{A_{n+1}}{C_{n+1}}\sum_{x=0}^{n} C_x + \frac{A_n}{E_n}\sum_{x=n+1}^{\infty} E_x}.$$

We start with the numerator. Using $\sum_{x=0}^{n} z^{-(x+1)} = \frac{z^{-(n+1)}-1}{1-z}$, we deduce that $\sum_{x=0}^{n} A_x = A_{n,s+1} - A_{-1,s+1} = A_{n,s+1}$, where $A_{n,s+1}$ corresponds to the building block A_n where s is replaced by $s+1$. Therefore, we have $\sum_{x=0}^{n} A_x \underset{a\to\infty}{\sim} \frac{a^n}{n!} + \frac{a^{n-1}(s+1)}{(n-1)!}$. Next, $\sum_{x=0}^{n}\frac{C_x}{C_{n+1}} \underset{a\to\infty}{\sim} \frac{n+1}{a} + \frac{n(n+1)}{a^2}$ and $A_{n+1} \underset{a\to\infty}{\sim} \frac{a^{n+1}}{(n+1)!} + \frac{a^n s}{n!}$. We thus deduce that $\frac{A_{n+1}}{C_{n+1}}\sum_{x=0}^{n} C_x - \sum_{x=0}^{n} A_x \underset{a\to\infty}{\sim} \frac{a^{n-1}s}{n!}$. Consider now the denominator of this expression. We have $\sum_{x=n+1}^{\infty} E_x =$

$\sum_{x=n+1}^{\infty} \frac{a^{x+s}}{\Gamma(x+1+s)} = \sum_{x=n+1+s}^{\infty} \frac{a^x}{\Gamma(x+1)} \underset{a\to\infty}{\sim} e^a$. Therefore, we deduce that $\frac{A_n}{E_n} \sum_{x=n+1}^{\infty} E_x \underset{a\to\infty}{\sim} e^a \frac{\Gamma(n+1+s)}{a^s n!}$. Finally, using $\frac{A_{n+1}}{C_{n+1}} \sum_{x=0}^{n} C_x \underset{a\to\infty}{\sim} \frac{a^n}{n!}$, we deduce the asymptotic expression of p_B^1.

Policy π_2. Recall that we have

$$p_B^2 = \frac{\sum_{x=0}^{n-2} \frac{E_x}{E_{-1}} + \frac{a\frac{E_{n-2}}{E_{-1}}}{a+s+n-1-n\frac{B_n}{B_{n-1}}} \sum_{x=n-1}^{\infty} \frac{B_x}{B_{n-1}}}{\sum_{x=-1}^{n-2} \frac{E_x}{E_{-1}} + \frac{a\frac{E_{n-2}}{E_{-1}}}{a+s+n-1-n\frac{B_n}{B_{n-1}}} \sum_{x=n-1}^{\infty} \frac{C_x}{C_{n-1}}}.$$

Combining $A_x \underset{a\to\infty}{\sim} \frac{a^x}{x!}$ with the expression of B_x in Proposition 2, we deduce that $\frac{B_x}{B_{x-1}} \underset{a\to\infty}{\sim} \frac{a}{x}$. Therefore, $a+s+n-1-n\frac{B_n}{B_{n-1}} \underset{a\to\infty}{\sim} s+n-1$. Using now that $\sum_{x=n-1}^{\infty} \frac{B_x}{B_{n-1}} = \frac{B_{n-2,s+1}}{B_{n-1}}$ leads to

$$\frac{a\frac{E_{n-2}}{E_{-1}}}{a+s+n-1-n\frac{B_n}{B_{n-1}}} \sum_{x=n-1}^{\infty} \frac{B_x}{B_{n-1}} \underset{a\to\infty}{\sim} \frac{a^n\Gamma(s)(n-1)}{s(s+n-1)\Gamma(s+n-1)}.$$

With the same asymptotic result as for p_B^1, we obtain

$$\frac{a\frac{E_{n-2}}{E_{-1}}}{a+s+n-1-n\frac{B_n}{B_{n-1}}} \sum_{x=n-1}^{\infty} \frac{C_x}{C_{n-1}} \underset{a\to\infty}{\sim} e^a \frac{(n-1)!\Gamma(s)}{\Gamma(n-1+s)(s+n-1)}.$$

From these expressions, we deduce that in the case where $n \geq 2$, the terms $\sum_{x=0}^{n-2} \frac{E_x}{E_{-1}}$ and $\sum_{x=-1}^{n-2} \frac{E_x}{E_{-1}}$ can be neglected in the numerator and denominator of p_B^2, which leads to the asymptotic expression of p_B^2. □

L Proof of Proposition 6

Proof. In this proposition, we assume that a and n are related through $\frac{n-a}{\sqrt{a}} = \beta$ and we determine asymptotic expressions for p_B^1 and p_B^2 as a and n tend to infinity.

Policy π_1. We express p_B^1 with

$$1 - p_B^1 = \frac{e^{-a}\sum_{x=0}^{n} \frac{a^x}{x!} - e^{-a} \frac{\sum_{x=0}^{n} A_x}{A_{n+1}} \frac{a^{n+1}}{(n+1)!}}{e^{-a}\sum_{x=0}^{n} \frac{a^x}{x!} + \frac{A_n}{A_{n+1}} \frac{a^{1-s}\Gamma(n+1+s)}{(n+1)!} e^{-a} \sum_{x=n+2}^{\infty} \frac{a^{x+s-1}}{\Gamma(x+s)}}.$$

We have $e^{-a}\sum_{x=0}^{n} \frac{a^x}{x!} = P\left(X_a < n + \frac{1}{2}\right) = P\left(\frac{X_a - a}{\sqrt{a}} \leq \beta + \frac{1}{2\sqrt{a}}\right)$, where X_a is a Poisson distribution with parameter a. Since X_a converges in distribution to a Normal distribution when a and n tend to infinity, we deduce that $e^{-a}\sum_{x=0}^{n}\frac{a^x}{x!}\underset{a,n\to\infty}{\sim}\Phi(\beta)+\frac{\Phi'(\beta)}{2\sqrt{a}}$, where Φ is the cdf of a Normal distribution with mean 0 and standard deviation 1.

Consider the random variable Y_a defined for $x\geq 0$ by

$$P(Y_a=x)=\frac{\frac{a^x\Gamma(s)}{\Gamma(s+x)}}{1+e^a a^{1-s}\gamma(s,a)}\text{ for } x\geq 0,$$

where $\gamma(s,a)$ is the incomplete gamma function defined as $\gamma(s,a)=\int_0^a e^{-t}t^{s-1}\mathrm{d}t$. We have

$$\gamma(s,a)=a^se^{-a}\sum_{k=0}^{\infty}\frac{a^k\Gamma(s)}{\Gamma(s+k+1)}=a^{s-1}e^{-a}\sum_{k=1}^{\infty}\frac{a^k\Gamma(s)}{\Gamma(s+k)}=a^{s-1}e^{-a}\left(\sum_{k=0}^{\infty}\frac{a^k\Gamma(s)}{\Gamma(s+k)}-1\right).$$

This proves that $\sum_{x=0}^{\infty}P(Y_a=x)=1$. Note that for $s=1$, Y_a has a Poisson distribution with parameter a. Since $\gamma(s,a)\underset{a\to\infty}{\sim}\Gamma(s)$ and $a^{s-1}e^a$ tends to infinity as a tends to infinity, we find an asymptotic distribution for Y_a given by

$$P(Y_a=x)\underset{a,n\to\infty}{\sim}e^{-a}\frac{a^{x+s-1}}{\Gamma(s+x)}\text{ for } x\geq 0,$$

as in the denominator of $1-p_B^1$. Therefore, we have

$$\begin{aligned}P(Y_a\geq n+2)&\underset{a,n\to\infty}{\sim}e^{-a}\sum_{x=n+2}^{\infty}\frac{a^{x+s-1}}{\Gamma(s+x)}\\&\underset{a,n\to\infty}{\sim}P\left(X_a>n+s+\frac{1}{2}\right)\underset{a,n\to\infty}{\sim}1-\Phi\left(\beta+\frac{s+1/2}{\sqrt{a}}\right)\underset{a,n\to\infty}{\sim}1-\Phi(\beta)-\Phi'(\beta)\frac{s+1/2}{\sqrt{a}}.\end{aligned}$$

This proves that $e^{-a}\sum_{x=n+2}^{\infty}\frac{a^{x+s-1}}{\Gamma(x+s)}\underset{a,n\to\infty}{\sim}1-\Phi\left(\beta+\frac{s+1/2}{\sqrt{a}}\right)$.

Using Stirling's formula, we get

$$a^{1-s}\frac{\Gamma(n+1+s)}{(n+1)!}\underset{a,n\to\infty}{\sim}\left(\frac{n+s}{n+1}\right)^{n+1+\frac{1}{2}}(n+s)^{s-1}e^{-s+1}a^{1-s}\underset{a,n\to\infty}{\sim}e^{\frac{\beta(s-1)}{\sqrt{a}}}\underset{a,n\to\infty}{\sim}1+\frac{\beta(s-1)}{\sqrt{a}}.$$

Using Equation (21), we show that

$$s\sum_{x=0}^{n}A_x=(n+1)A_{n+1}-aA_n.$$

Therefore, we obtain

$$e^{-a}\frac{\sum_{x=0}^{n} A_x}{A_{n+1}}\frac{a^{n+1}}{(n+1)!} = e^{-a}\frac{a^n}{n!}\frac{a}{s}\left(1-\frac{a}{n+1}\frac{A_n}{A_{n+1}}\right).$$

Using again Stirling's formula leads to

$$e^{-a}\frac{a^n}{n!} \underset{a,n\to\infty}{\sim} \frac{e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi n}} = \frac{\Phi'(\beta)}{\sqrt{n}}.$$

There remains to determine an equivalent expression for the ratio $\frac{A_n}{A_{n+1}}$. For this purpose, we change the variable z in the integral definition of A_x by $z = 1 - \frac{u}{\sqrt{n}}$. We thus obtain

$$A_n \underset{a,n\to\infty}{\sim} \frac{1}{2\pi i} n^{\frac{s}{2}-\frac{1}{2}} e^a \int_{\zeta_u} u^{-s} e^{\beta u} e^{\frac{u^2}{2}} \, \mathrm{d}\, u = \frac{n^{\frac{s}{2}-\frac{1}{2}} e^a}{\sqrt{2\pi}} e^{-\frac{\beta^2}{4}} P^c_{-s}(-\beta),$$

where $P^c_x(y)$ is the parabolic cylinder function of index x and argument y and the approximating contour ζ_u is a vertical contour on which $Re(u) > 0$. Next, we also have

$$A_{n+1} - A_n = \frac{1}{2i\pi}\int_{\zeta_1} z^{-(n+2)} e^{az}(1-z)^{1-s}\mathrm{d}z \underset{a,n\to\infty}{\sim} \frac{n^{\frac{s}{2}-1} e^a}{\sqrt{2\pi}} e^{-\frac{\beta^2}{4}} P^c_{1-s}(-\beta).$$

This leads to

$$\frac{A_n}{A_{n+1}} \underset{a,n\to\infty}{\sim} 1 - \frac{P^c_{1-s}(-\beta)}{\sqrt{n} P^c_{-s}(-\beta)}.$$

We thus deduce the approximated expressions of p^1_B.

Policy π_2. Using flow conservation, we have $a = \sum_{x=1}^{\infty} x(p^2_x + q^2_x) + sp^2_B$. Therefore, we express p^2_B as

$$p^2_B = \frac{a}{s} - \frac{1}{s}\frac{e^{-a}\sum_{x=-1}^{n-2}\frac{xa^{x+s}}{\Gamma(x+1+s)}\frac{a^{-s}\Gamma(n+s)}{(n-1)!} + \frac{n-1+s}{a+s+n-1-n\frac{B_n}{B_{n-1}}}\sum_{x=n-1}^{\infty}\frac{xa^x}{x!}e^{-a}}{e^{-a}\sum_{x=-1}^{n-2}\frac{a^{x+s}}{\Gamma(x+1+s)}\frac{a^{-s}\Gamma(n+s)}{(n-1)!} + \frac{n-1+s}{a+s+n-1-n\frac{B_n}{B_{n-1}}}\sum_{x=n-1}^{\infty}\frac{a^x}{x!}e^{-a}}$$

Using Stirling's formula, we have

$$\frac{a^{-s}\Gamma(n+s)}{(n-1)!} \underset{a,n\to\infty}{\sim} 1 + s\frac{\beta}{\sqrt{a}} + \frac{(s-1)(\beta^2(s+1)+s)}{2a}.$$

Next,

$$e^{-a}\sum_{x=-1}^{n-2}\frac{a^{x+s}}{\Gamma(x+1+s)} \underset{a,n\to\infty}{\sim} P\left(s-\frac{3}{2}<X_a<s+n-\frac{3}{2}\right) \underset{a,n\to\infty}{\sim} \Phi(\beta)+\frac{s-\frac{3}{2}}{\sqrt{a}}\Phi'(\beta)+\frac{\left(s-\frac{3}{2}\right)^2}{a}\frac{\Phi''(\beta)}{2}.$$

Next, using $x = x + s - s$, we find that

$$\begin{aligned}
e^{-a}\sum_{x=1}^{n-2}\frac{xa^{x+s}}{\Gamma(x+1+s)} &= e^{-a}a\sum_{x=0}^{n-3}\frac{a^{x+s}}{\Gamma(x+1+s)} - se^{-a}\sum_{x=1}^{n-2}\frac{a^{x+s}}{\Gamma(x+1+s)}\\
&\underset{a,n\to\infty}{\sim} aP\left(s-\frac{1}{2}<X_a<s+n-\frac{5}{2}\right) - sP\left(s-\frac{3}{2}<X_a<s+n-\frac{3}{2}\right)\\
&\underset{a,n\to\infty}{\sim} a\left(\Phi(\beta)+\frac{s-\frac{5}{2}}{\sqrt{a}}\Phi'(\beta)+\frac{\left(s-\frac{5}{2}\right)^2}{a}\frac{\Phi''(\beta)}{2}\right) - s\left(\Phi(\beta)+\frac{s-\frac{3}{2}}{\sqrt{a}}\Phi'(\beta)+\frac{\left(s-\frac{3}{2}\right)^2}{a}\frac{\Phi''(\beta)}{2}\right).
\end{aligned}$$

We also have

$$\sum_{x=n-1}^{\infty} e^{-a}\frac{a^x}{x!} = 1 - P\left(X_a < n-\frac{3}{2}\right) \underset{a,n\to\infty}{\sim} 1-\Phi(\beta)+\frac{3}{2\sqrt{a}}\Phi'(\beta)-\frac{9}{8a}\Phi''(\beta),$$

and

$$\sum_{x=n-1}^{\infty} xe^{-a}\frac{a^x}{x!} = a\sum_{x=n-2}^{\infty} e^{-a}\frac{a^x}{x!} = a\left(1-P\left(X_a<n-\frac{5}{2}\right)\right) \underset{a,n\to\infty}{\sim} a\left(1-\Phi(\beta)+\frac{5}{2\sqrt{a}}\Phi'(\beta)-\frac{25}{8a}\Phi''(\beta)\right).$$

From Lemma 3, we deduce that $\frac{B_n}{B_{n-1}} \underset{a,n\to\infty}{\sim} \frac{a}{n+s}$. Therefore, we have

$$\frac{n-1+s}{a+s+n-1-n\frac{B_n}{B_{n-1}}} \underset{a,n\to\infty}{\sim} 1.$$

Using the above results, we find that

$$p_B^2 \underset{a,n\to\infty}{\sim} \Phi(\beta)+\beta\Phi'(\beta)+\Phi''(\beta).$$

Finally, the relation $\Phi''(\beta) = -\beta\Phi'(\beta)$ leads to the result. □