Online Supplement to "Optimal Scheduling in Call Centers with a Callback Option"

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Performance Evaluation, 95:1-40, 2016

This is an online supplement to the main paper, with the same title.

1 Proof of Corollary 1

We first rewrite the value functions in the single server case. So as to simplify the presentation of the proof, we redefine the states as follows; the parameter z denotes the state of the agent (0: idling, 1: busy with an inbound or an outbound call), x is the number of inbounds in queue 1 and y is the number of outbounds in queue 2. We have for $n \geq 0$,

$$
\begin{aligned}
&U_{n+1}(0,0,y) = \gamma_2 y + \lambda V_n(1,0,y) + (1-\lambda)V_n(0,0,y), \text{ for } y \geq 0\\
&U_{n+1}(1,x,y) = \gamma_1 x + \gamma_2 y + \lambda \big(\mathbf{1}_{(0\leq x<k)} \left((1-\alpha_x)V_n(1,x+1,y) + \alpha_x(V_n(1,x,y)+\gamma_3)\right)\\
&+\mathbf{1}_{(k\leq x<N)} \left(q_x V_n(1,x,y+1) + \alpha_x(V_n(1,x,y)+\gamma_3) + (1-q_x-\alpha_x)V_n(1,x+1,y)\right)\\
&+\mathbf{1}_{(x=N)}(q_{N-1}V_n(1,x,y+1) + (1-q_{N-1})(V_n(1,x,y)+\gamma_3))\big)\\
&+ \beta x(V_n(1,x-1,y)+\gamma_3) + \mu \left(\mathbf{1}_{(x=0)}V_n(0,0,y) + \mathbf{1}_{(x>0)}V_n(1,x-1,y)\right)\\
&+ (1-\lambda-\beta x-\mu)\, V_n(1,x,y), \text{ for } x,y \geq 0,
\end{aligned}
$$

with $V_{n+1}(0,0,y) = \min(U_{n+1}(1,0,y-1), U_{n+1}(0,0,y))$ for $y > 0$, and $V_{n+1}(z,x,y) = U_{n+1}(z,x,y)$ in the remaining cases. We choose $V_0(z,x,y) = U_0(z,x,y) = 0$, for $z = 0,1$ and $x,y \geq 0$.

We define a class of functions $\mathcal{F}$ from $\{0,1\} \times \mathbb{N}^2$ to $\mathbb{R}$ as follows: $f \in \mathcal{F}$ if for $z = 0,1$ and $x,y \geq 0$, we have

$$f(1,x+1,y) \geq f(1,x,y), \tag{1}$$

$$f(1,0,y) \geq f(0,0,y) \tag{2}$$

$$f(1,x,y+1) \geq f(1,x,y), \tag{3}$$

$$f(0,0,y+1) \geq f(0,0,y), \tag{4}$$

$$f(1,x,y) + f(1,x+1,y+1) \geq f(1,x+1,y) + f(1,x,y+1), \tag{5}$$

$$f(0,0,y) + f(1,0,y+1) \geq f(1,0,y) + f(0,0,y+1), \tag{6}$$

$$f(1,x,y+2) + f(1,x+1,y) \geq f(1,x,y+1) + f(1,x+1,y+1), \tag{7}$$

$$f(0,0,y+2) + f(1,0,y) \geq f(0,0,y+1) + f(1,0,y+1). \tag{8}$$

Relations (1) and (3) define a class of increasing functions in x and in y. Relation (5) is known as supermodularity. By summing up Relations (5) and (7) we obtain $f(1,x,y) + f(1,x,y+2) \geq 2f(1,x,y+1)$, and by summing up Relations (6) and (8) we obtain $f(0,0,y) + f(0,0,y+2) \geq 2f(0,0,y+1)$. Thus if $f \in \mathcal{F}$, then f is convex in y. Relations (7) and (8) means that the function $f(z,x,y+1) - f(z,x+1,y)$ is increasing in y for $z = 0,1$. Note that only Relation (8) proves the optimality of the threshold policy. However, all the remaining relations have to be proven together to show the propagation of Relation (8).

To simplify the presentation, we denote by "serve" the decision action to serve an outbound call, and by "keep" the decision action to keep an outbound call in queue 2. The proof of the optimality of the threshold policy reduces to show that Relation (8) is true for U_n, $n \geq 0$.
In what follows, we prove by induction on n that both V_n and U_n are in $\mathcal{F}$. We divide the proof into the following 5 steps:

- **Step 1.** We prove that $V_0, U_0 \in \mathcal{F}$.
- **Step 2.** We prove that if for a given n, $U_n \in \mathcal{F}$, then $V_n \in \mathcal{F}$.
- **Step 3.** We prove that the cost term $G(z,x,y) = \gamma_1 x + \gamma_2 y$ is in $\mathcal{F}$ for $z = 0,1$ and $x, y \geq 0$.
- **Step 4.** We prove that if for a given n, $V_n \in \mathcal{F}$, then the arrival term defined by

$$\begin{aligned} A_n(1,x,y) = &\mathbf{1}_{(0 \leq x < k)} \left((1-\alpha_x)V_n(1,x+1,y) + \alpha_x(V_n(1,x,y)+\gamma_3)\right) \\ &+\mathbf{1}_{(k \leq x < N)} \left(q_x V_n(1,x,y+1) + \alpha_x(V_n(1,x,y)+\gamma_3) + (1-q_x-\alpha_x)V_n(1,x+1,y)\right) \\ &+ \mathbf{1}_{(x=N)}(q_{N-1}V_n(1,x,y+1) + (1-q_{N-1})(V_n(1,x,y)+\gamma_3)) \end{aligned}$$

  for $x, y \geq 0$ and $A_n(0,0,y) = V_n(1,0,y)$ for $y \geq 0$, is also in $\mathcal{F}$.

- **Step 5.** We prove that if for a given n, $V_n \in \mathcal{F}$, then the departure term defined by

$$D_n(1,x,y) = \beta x(V_n(1,x-1,y)+\gamma_3) + \mu\left(\mathbf{1}_{(x=0)}V_n(0,0,y) + \mathbf{1}_{(x>0)}V_n(1,x-1,y)\right) \\ + (1-\lambda-\beta x-\mu)\,V_n(1,x,y)$$

for $x, y \geq 0$ and $D_n(0,0,y) = (1-\lambda)V_n(0,0,y)$ for $y \geq 0$, is also in $\mathcal{F}$.

The proofs for the previous five steps are given below.

Step 1. For $x, y \geq 0$ and $z = 0, 1$, $V_0(z,x,y) = U_0(z,x,y) = 0$. Then $V_0, U_0 \in \mathcal{F}$.

Step 2. Assume that for a given $n \geq 0$, $U_n \in \mathcal{F}$. We only consider the non trivial cases where $z = 0$ and $y > 0$. In the other cases, we have $U_n = V_n$. Therefore we only need to prove Relations (2), (4), (6) and (8).

- For Relations (2) and (4), we have

$$V_n(0,0,y) \leq U_n(0,0,y), \tag{9}$$

$$V_n(0,0,y) \leq U_n(1,0,y-1). \tag{10}$$

Since no action can be chosen if the server is busy, $V_n(1,0,y) = U_n(1,0,y)$. Combining Inequality (9) with Relation (2) for U_n leads to $V_n(0,0,y) \leq V(1,0,y)$ and proves Relations (2) for V_n.
If "keep" is optimal in $(0,0,y{+}1)$, then $V_n(0,0,y{+}1) = U_n(0,0,y{+}1)$. Combining Inequality (9) with Relation (4) for U_n leads to $V_n(0,0,y) \leq V(0,0,y+1)$. If "serve" is optimal in $(0,0,y+1)$, then $V_n(0,0,y+1) = U_n(1,0,y)$. Combining Inequality (10) with Relation (3) for U_n leads to $V_n(0,0,y) \leq V(0,0,y{+}1)$. Therefore in all cases, Relations (2) and (4) hold for V_n.

- For Relation (6), we have

$$V_n(1,0,y) + V_n(0,0,y+1) \leq U_n(1,0,y) + U_n(0,0,y+1) \text{ for } y \geq 0, \tag{11}$$

$$V_n(1,0,y) + V_n(0,0,y+1) \leq 2U_n(1,0,y) \text{ for } y \geq 0. \tag{12}$$

If "keep" is the optimal action in state $(0,0,y)$, for $y > 0$, then $V_n(0,0,y) + V_n(1,0,y+1) = U_n(0,0,y) + U_n(1,0,y+1)$. Thus, combining Relation (6) for U_n and Inequality (11) proves Relation (6) for V_n, for $y \geq 0$. If "serve" is the optimal action in state $(0,0,y)$, for $y > 0$, then $V_n(0,0,y) + V_n(1,0,y+1) = U_n(1,0,y-1) + U_n(1,0,y+1)$. Combining the convexity in y of U_n and Inequality (12) proves Relation (6) for V_n, for $y > 0$. In all cases, Relation (6) then holds for V_n.

- For Relation (8), we have

$$V_n(0,0,y+1) + V_n(1,0,y+1) \leq U_n(0,0,y+1) + U_n(1,0,y+1) \text{ for } y \geq 0, \tag{13}$$

$$V_n(0,0,y+1) + V_n(1,0,y+1) \leq U_n(1,0,y) + U_n(1,0,y+1) \text{ for } y \geq 0. \tag{14}$$

If "keep" is the optimal action in states $(0,0,y+2)$, for $y \geq 0$, then $V_n(0,0,y+2) + V_n(1,0,y) = U_n(0,0,y+$

2) + $U_n(1,0,y)$. Combining next Relation (8) for U_n and Inequality (13) proves Relation (8) for V_n, for $y \geq 0$. If "serve" is the optimal action in state $(0,0,y+2)$, for $y \geq 0$, then $V_n(0,0,y+2) + V_n(1,0,y) = U_n(1,0,y+1) + U_n(1,0,y)$. Inequality (14) proves Relation (8) for V_n, for $y \geq 0$. Finally we deduce for all cases that Relation (8) is true for V_n.

Step 3. This step is easy to prove and directly follows from Koole (2007) page 33.

Step 4. Assume that $V_n \in \mathcal{F}$, for a given $n \geq 0$. We now show that $A_n \in \mathcal{F}$

In Relation (4) we have $x = 0$ and the arrival of a new call has the same effect on each term of the relation. Since the transition rates are constant, the induction from V_n to A_n is direct (see Koole (2007) page 35)

The other relations have to be shown to prove the induction from V_n to A_n. For Relations (1), (3), (5) and (7), the case $x < k-1$ is a simplification of the case $k \leq x < N-1$ since the possibility of going to queue 2 is not considered, we therefore only show the case $k \leq x < N-1$.

- For Relation (1):

If $x = k-1$, then

$$\begin{aligned}
A_n(1,x+1,y) - A_n(1,x,y) &= q_k V_n(1,x+1,y+1) + \alpha_k(V_n(1,x+1,y) + \gamma_3) + (1-\alpha_k - q_k)V_n(1,x+2,y) \\
&\quad - (1-\alpha_{k-1})V_n(1,x+1,y) - \alpha_{k-1}(V_n(1,x,y) + \gamma_3) \\
&= q_k\left(V_n(1,x+1,y+1) - V_n(1,x+1,y)\right) + \alpha_{k-1}(V_n(1,x+1,y) - V_n(1,x,y)) \\
&\quad + (1-\alpha_k - q_k)(V_n(1,x+2,y) - V_n(1,x+1,y)) + \gamma_3(\alpha_k - \alpha_{k-1}) \geq 0,
\end{aligned}$$

since V_n is increasing in x and in y.

If $k \leq x < N-1$, then

$$\begin{aligned}
A_n(1,x+1,y) - A_n(1,x,y) &= q_{x+1}V_n(1,x+1,y+1) + \alpha_{x+1}(V_n(1,x+1,y) + \gamma_3) + (1-\alpha_{x+1} - q_{x+1})V_n(1,x+2,y) \\
&\quad - q_x V_n(1,x,y+1) - \alpha_x(V_n(1,x,y) + \gamma_3) - (1-\alpha_x - q_x)V_n(1,x+1,y) \\
&= q_x\left(V_n(1,x+1,y+1) - V_n(1,x+1,y)\right) + (q_{x+1} - q_x)V_n(1,x+1,y+1) \\
&\quad + \alpha_x(V_n(1,x+1,y) - V_n(1,x,y)) + (\alpha_{x+1} - \alpha_x)V_n(1,x+1,y) + \gamma_3(\alpha_{x+1} - \alpha_x) \\
&\quad + (1-\alpha_{x+1} - q_{x+1})(V_n(1,x+2,y) - V_n(1,x+1,y)) + (\alpha_x + q_x - \alpha_{x+1} - q_{x+1})V_n(1,x+1,y) \\
&\geq (q_{x+1} - q_x)(V_n(1,x+1,y+1) - V_n(1,x+1,y)) \geq 0,
\end{aligned}$$

since V_n is increasing in y and q_x is increasing in x.

If $x = N-1$, then

$$\begin{aligned}
A_n(1,x+1,y) - A_n(1,x,y) &= q_x V_n(1,x+1,y+1) + (1-q_x)(V_n(1,x+1,y) + \gamma_3) \\
&\quad - q_x V_n(1,x,y+1) - \alpha_x(V_n(1,x,y) + \gamma_3) - (1-\alpha_x - q_x)V_n(1,x+1,y) \\
&= \alpha_x(V_n(1,x+1,y+1) - V_n(1,x,y)) + q_x(V_n(1,x+1,y+1) - V_n(1,x,y+1)) \\
&\quad + \gamma_3(1-q_x-\alpha_x) \geq 0,
\end{aligned}$$

since V_n is increasing in x and in y. Finally for all cases, Relation (1) is true for A_n.

- For Relation (2), we have

$$\begin{aligned}A_n(1,0,y)-A_n(0,0,y)&=(1-\alpha_0)V_n(1,1,y)+\alpha_0(V_n(1,0,y)+\gamma_3)-V_n(1,0,y)\\&=(1-\alpha_0)(V_n(1,1,y)-V_n(1,0,y))+\alpha_0\gamma_3\geq 0,\end{aligned}$$

since Relation (1) is true for V_n. Hence, Relation (2) is true for A_n.

The propagation of Relation (3) through the arrival operator is straightforward.

For the following relations, we do not write the terms in γ_3 since they simplify in the considered differences.

- For Relation (5):
If $x=k-1$, then

$$\begin{aligned}&A_n(1,x,y)+A_n(1,x+1,y+1)-A_n(1,x,y+1)-A_n(1,x+1,y)\\&=\alpha_{k-1}V_n(1,x,y)+(1-\alpha_{k-1})V_n(1,x+1,y)+q_kV_n(1,x+1,y+2)+\alpha_kV_n(1,x+1,y+1)+(1-\alpha_k-q_k)V_n(1,x+2,y+1)\\&-\alpha_{k-1}V_n(1,x,y+1)-(1-\alpha_{k-1})V_n(1,x+1,y+1)-q_kV_n(1,x+1,y+1)-\alpha_kV_n(1,x+1,y)-(1-\alpha_k-q_k)V_n(1,x+2,y)\\&=\alpha_{k-1}(V_n(1,x+1,y+1)+V_n(1,x,y)-V_n(1,x+1,y)-V_n(1,x,y+1))\\&+q_k(V_n(1,x+1,y+2)+V_n(1,x+2,y)-V_n(1,x+2,y+1)-V_n(1,x+1,y+1))\\&+(1-\alpha_k)(V_n(1,x+2,y+1)+V_n(1,x+1,y)-V_n(1,x+1,y+1)-V_n(1,x+2,y)).\end{aligned}$$

The term proportional to α_{k-1} is positive since Relation (5) is true for V_n, the term proportional to q_k is positive since Relation (7) is true for V_n, the term proportional to $1-\alpha_k$ is positive since Relation (5) is true for V_n. Hence Relation (5) is true for A_n for $x=k-1$.
If $k\leq x<N-1$, then

$$\begin{aligned}&A_n(1,x,y)+A_n(1,x+1,y+1)-A_n(1,x,y+1)-A_n(1,x+1,y)\\&=q_xV_n(1,x,y+1)+\alpha_xV_n(1,x,y)+(1-q_x-\alpha_x)V_n(1,x+1,y)\\&+q_{x+1}V_n(1,x+1,y+2)+\alpha_{x+1}V_n(1,x+1,y+1)+(1-q_{x+1}-\alpha_{x+1})V_n(1,x+2,y+1)\\&-q_xV_n(1,x,y+2)-\alpha_xV_n(1,x,y+1)-(1-q_x-\alpha_x)V_n(1,x+1,y+1)\\&-q_{x+1}V_n(1,x+1,y+1)-\alpha_{x+1}V_n(1,x+1,y)-(1-q_{x+1}-\alpha_{x+1})V_n(1,x+2,y)\\&=q_x(V_n(1,x,y+1)+V_n(1,x+1,y+2)-V_n(1,x,y+2)-V_n(1,x+1,y+1))\\&+\alpha_x(V_n(1,x,y)+V_n(1,x+1,y+1)-V_n(1,x,y+1)-V_n(1,x+1,y))\\&+(1-\alpha_{x+1}-q_{x+1})(V_n(1,x+1,y)+V_n(1,x+2,y+1)-V_n(1,x+1,y+1)-V_n(1,x+2,y))\\&+(q_{x+1}-q_x)(V_n(1,x+1,y+2)+V_n(1,x+1,y)-2V_n(1,x+1,y+1)).\end{aligned}$$

The terms proportional to q_x, α_x and $1-q_{x+1}-\alpha_{x+1}$ are positive because Relation (5) is true for V_n. The term proportional to $q_{x+1}-q_x$ is also positive because V_n is convex in y. Thus, Relation (5) is true for A_n for $k\leq x<N-1$.
If $x=N-1$, then

$$\begin{aligned}&A_n(1,x,y)+A_n(1,x+1,y+1)-A_n(1,x,y+1)-A_n(1,x+1,y)\\&=q_xV_n(1,x,y+1)+\alpha_xV_n(1,x,y)+(1-q_x-\alpha_x)V_n(1,x+1,y)+q_xV_n(1,x+1,y+2)+(1-q_x)V_n(1,x+1,y+1)\\&-q_xV_n(1,x,y+2)-\alpha_xV_n(1,x,y+1)-(1-q_x-\alpha_x)V_n(1,x+1,y+1)-q_xV_n(1,x+1,y+1)-(1-q_x)V_n(1,x+1,y)\\&=q_x(V_n(1,x,y+1)+V_n(1,x+1,y+2)-V_n(1,x,y+2)-V_n(1,x+1,y+1))\\&+\alpha_x(V_n(1,x,y)+V_n(1,x+1,y+1)-V_n(1,x,y+1)-V_n(1,x+1,y)).\end{aligned}$$

The terms proportional to q_x and α_x are positive since Relation (5) is true for V_n. Hence Relation (5) is true for A_n for $x = N-1$.

- For Relation (6), we have for $y \geq 0$,

$$\begin{aligned}
&A_n(0,0,y) + A_n(1,0,y+1) - A_n(1,0,y) - A_n(0,0,y+1) \\
&= V_n(1,0,y) + \alpha_0 V_n(1,0,y+1) + (1-\alpha_0)V_n(1,1,y+1) - \alpha_0 V_n(1,0,y) - (1-\alpha_0)V_n(1,1,y) - V_n(1,0,y+1) \\
&= (1-\alpha_0)(V_n(1,0,y) + V_n(1,1,y+1) - V_n(1,1,y) - V_n(1,0,y+1)) \geq 0,
\end{aligned}$$

because Relation (5) is true for V_n. Hence Relation (6) holds for A_n.

- For Relation (7):

If $x < k-1$, the transition rates are constant and the induction from V_n to A_n is straightforward.
If $x = k-1$, we may write

$$\begin{aligned}
&A_n(1,x,y+2) + A_n(1,x+1,y) - A_n(1,x,y+1) - A_n(1,x+1,y+1) \\
&= \alpha_{k-1}V_n(1,x,y+2) + (1-\alpha_{k-1})V_n(1,x+1,y+2) + q_k V_n(1,x+1,y+1) + \alpha_k V_n(1,x+1,y) + (1-\alpha_k-q_k)V_n(1,x+2,y) \\
&- \alpha_{k-1}V_n(1,x,y+1) - (1-\alpha_{k-1})V_n(1,x+1,y+1) - q_k V_n(1,x+1,y+2) - \alpha_k V_n(1,x+1,y+1) \\
&- (1-\alpha_k-q_k)V_n(1,x+2,y+1) \\
&= \alpha_{k-1}(V_n(1,x,y+2) + V_n(1,x+1,y+1) - V_n(1,x+1,y+2) - V_n(1,x,y+1)) \\
&+ \alpha_k(V_n(1,x+1,y) + V_n(1,x+2,y+1) - V_n(1,x+2,y) - V_n(1,x+1,y+1)) \\
&+ q_k(V_n(1,x+1,y+1) + V_n(1,x+2,y+1) - V_n(1,x+2,y) - V_n(1,x+1,y+2)) \\
&+ V_n(1,x+2,y) + V_n(1,x+1,y+2) - V_n(1,x+1,y+1) - V_n(1,x+2,y+1) \\
&= (\alpha_k - \alpha_{k-1})(V_n(1,x+1,y) + V_n(1,x+1,y+2) - 2V_n(1,x+1,y+1)) \\
&+ \alpha_{k-1}(V_n(1,x,y+2) + V_n(1,x+1,y) - V_n(1,x,y+1) - V_n(1,x+1,y+1)) \\
&(1-q_k-\alpha_k)(V_n(1,x+2,y) + V_n(1,x+1,y+2) - V_n(1,x+1,y+1) - V_n(1,x+2,y+1)).
\end{aligned}$$

The term proportional to $\alpha_k - \alpha_{k-1}$ is positive since V_n is convex in y, the term proportional to α_{k-1} is positive since Relation (7) is true for V_n, the term proportional to $1 - q_k - \alpha_k$ is positive since Relation (7) is true for V_n. Relation (7) holds therefore for A_n for $x = k-1$.
If $k \leq x < N-1$, then

$$\begin{aligned}
&A_n(1,x,y+2) + A_n(1,x+1,y) - A_n(1,x,y+1) - A_n(1,x+1,y+1) \\
&= q_x V_n(1,x,y+3) + \alpha_x V_n(1,x,y+2) + (1-q_x-\alpha_x)V_n(1,x+1,y+2) \\
&+ q_{x+1}V_n(1,x+1,y+1) + \alpha_{x+1}V_n(1,x+1,y) + (1-q_{x+1}-\alpha_{x+1})V_n(1,x+2,y) \\
&- q_x V_n(1,x,y+2) - \alpha_x V_n(1,x,y+1) - (1-q_x-\alpha_x)V_n(1,x+1,y+1) \\
&- q_{x+1}V_n(1,x+1,y+2) - \alpha_{x+1}V_n(1,x+1,y+1) - (1-q_{x+1}-\alpha_{x+1})V_n(1,x+2,y+1) \\
&= q_x(V_n(1,x,y+3) + V_n(1,x+1,y+1) - V_n(1,x+1,y+2) - V_n(1,x,y+2)) \\
&+ \alpha_x(V_n(1,x,y+2) + V_n(1,x+1,y) - V_n(1,x+1,y+1) - V_n(1,x,y+1)) \\
&+ (1-\alpha_{x+1}-q_{x+1})(V_n(1,x+1,y+2) + V_n(1,x+2,y) - V_n(1,x+1,y+1) - V_n(1,x+2,y+1)) \\
&+ (\alpha_{x+1}-\alpha_x)(V_n(1,x+1,y+2) + V_n(1,x+1,y) - 2V_n(1,x+1,y+1)).
\end{aligned}$$

The terms proportional to q_x, α_x and $1 - q_{x+1} - \alpha_{x+1}$ are positive since Relation (7) is true for V_n, the term proportional to $\alpha_{x+1} - \alpha_x$ is also positive since V_n is convex in y. Hence, Relation (7) is true for A_n for $k \leq x < N-1$.

If $x = N-1$, then

$$A_n(1,x,y+2) + A_n(1,x+1,y) - A_n(1,x,y+1) - A_n(1,x+1,y+1)$$
$$= q_x V_n(1,x,y+3) + \alpha_x V_n(1,x,y+2) + (1-q_x-\alpha_x)V_n(1,x+1,y+2) + q_x V_n(1,x+1,y+1) + (1-q_x)V_n(1,x+1,y)$$
$$- q_x V_n(1,x,y+2) - \alpha_x V_n(1,x,y+1) - (1-q_x-\alpha_x)V_n(1,x+1,y+1) - q_x V_n(1,x+1,y+2) - (1-q_x)V_n(1,x+1,y+1)$$
$$= q_x(V_n(1,x,y+3) + V_n(1,x+1,y+1) - V_n(1,x+1,y+2) - V_n(1,x,y+2))$$
$$+ \alpha_x(V_n(1,x,y+2) + V_n(1,x+1,y) - V_n(1,x,y+1) - V_n(1,x+1,y+1))$$
$$+ (1-q_x-\alpha_x)(V_n(1,x+1,y+2) + V_n(1,x+1,y) - 2V_n(1,x+1,y+1)).$$

The terms proportional to q_x and α_x are positive since Relation (7) is true for V_n, the term proportional to $1-q_x-\alpha_x$ is also positive since V_n is convex in y. Hence Relation (7) is true for A_n for $x = N-1$.

- For Relation (8), we have

$$A_n(0,0,y+2) + A_n(1,0,y) - A_n(0,0,y+1) - A_n(1,0,y+1)$$
$$= V_n(1,0,y+2) + \alpha_0 V_n(1,0,y) + (1-\alpha_0)V_n(1,1,y) - V_n(1,0,y+1) - \alpha_0 V_n(1,0,y+1) - (1-\alpha_0)V_n(1,1,y+1)$$
$$= V_n(1,0,y+2) + V_n(1,1,y) - V_n(1,0,y+1) - V_n(1,1,y+1)$$
$$+ \alpha_0(V_n(1,0,y) + V_n(1,1,y+1) - V_n(1,0,y+1) - V_n(1,1,y)).$$

The first term is positive since Relation (7) is true for V_n, the term proportional to α_0 is also positive since Relation (5) is true for V_n. So, Relation (8) holds for A_n.

Step 5. Assume that $V_n \in \mathcal{F}$, for a given $n \geq 0$. We now show that $D_n \in \mathcal{F}$.

- For Relation (1):

If $x = 0$, then

$$D_n(1,1,y) - D_n(1,0,y) = \beta V_n(1,0,y) + \beta\gamma_3 + \mu(V_n(1,0,y) - V_n(0,0,y))$$
$$+ (1-\lambda-\beta-\mu)(V_n(1,1,y) - V_n(1,0,y)) - \beta V_n(1,0,y) \geq 0,$$

because V_n is increasing in x and since Relation (2) is true for V_n.

If $x > 0$, then

$$D_n(1,x+1,y) - D_n(1,x,y) = \beta x(V_n(1,x,y) - V_n(1,x-1,y)) + \beta\gamma_3 + \beta V_n(1,x,y) + \mu(V_n(1,x,y) - V_n(1,x-1,y))$$
$$+ (1-\lambda-\beta(x+1)-\mu)(V_n(1,x+1,y) - V_n(1,x,y)) - \beta V_n(1,x,y) \geq 0,$$

since V_n is increasing in x. Hence Relation (1) is true for D_n.

- For Relation (2), we have

$$D_n(1,0,y) - D_n(0,0,y) = \mu V_n(0,0,y) + (1-\lambda-\mu)(V_n(1,0,y) - V_n(0,0,y)) - \mu V_n(0,0,y) \geq 0.$$

Then, Relation (2) holds for D_n.

- For Relation (3):

If $x \geq 0$, then

$$D_n(1,x,y+1) - D_n(1,x,y) = \beta x(V_n(1,x-1,y+1) - V_n(1,x-1,y))$$
$$+ \mu\mathbf{1}_{(x=0)}(V_n(0,0,y+1) - V_n(0,0,y)) + \mu\mathbf{1}_{(x>0)}(V_n(1,x-1,y+1) - V_n(1,x-1,y))$$
$$+ (1-\lambda-\beta x-\mu)(V_n(1,x,y+1) - V_n(1,x,y)) \geq 0,$$

since V_n is increasing in y. Thus, Relation (3) is true for D_n.

- Relation (4) is obviously also true.

- For Relation (5):

If $x, y \geq 0$, then

$$
\begin{aligned}
& D_n(1,x,y) + D_n(1,x+1,y+1) - D_n(1,x+1,y) - D_n(1,x,y+1) \\
& = \beta x (V_n(1,x-1,y) + V_n(1,x,y+1) - V_n(1,x,y) - V_n(1,x-1,y+1)) + \beta(V_n(1,x,y+1) - V_n(1,x,y)) \\
& + \mu \mathbf{1}_{(x=0)}(V_n(0,0,y) + V_n(1,0,y+1) - V_n(1,0,y) - V_n(0,0,y+1)) \\
& + \mu \mathbf{1}_{(x>0)}(V_n(1,x-1,y) + V_n(1,x,y+1) - V_n(1,x,y) - V_n(1,x-1,y+1)) \\
& + (1 - \lambda - \beta(x+1) - \mu)(V_n(1,x,y) + V_n(1,x+1,y+1) - V_n(1,x+1,y) - V_n(1,x,y+1)) \\
& + \beta(V_n(1,x,y) - V_n(1,x,y+1)) \geq 0,
\end{aligned}
$$

since Relations (5) and (6) are true for V_n.

- For Relation (6), we have for $y \geq 0$,

$$
\begin{aligned}
& D_n(0,0,y) + D_n(1,0,y+1) - D_n(1,0,y) - D_n(0,0,y+1) \\
& = \mu(V_n(0,0,y+1) - V_n(0,0,y)) + (1 - \lambda - \mu)(V_n(0,0,y) + V_n(1,0,y+1) - V_n(1,0,y) - V_n(0,0,y+1)) \\
& + \mu(V_n(0,0,y) - V_n(0,0,y+1)) \geq 0,
\end{aligned}
$$

since Relation (6) is true for V_n.

- For Relation (7):

If $x, y \geq 0$, then

$$
\begin{aligned}
& D_n(1,x,y+2) + D_n(1,x+1,y) - D_n(1,x,y+1) - D_n(1,x+1,y+1) \\
& = \beta x (V_n(1,x-1,y+2) + V_n(1,x,y) - V_n(1,x-1,y+1) - V_n(1,x,y+1)) + \beta(V_n(1,x,y) - V_n(1,x,y+1)) \\
& + \mu \mathbf{1}_{(x=0)}(V_n(0,0,y+2) + V_n(1,0,y) - V_n(0,0,y+1) - V_n(1,0,y+1)) \\
& + \mu \mathbf{1}_{(x>0)}(V_n(1,x-1,y+2) + V_n(1,x,y) - V_n(1,x-1,y+1) - V_n(1,x,y+1)) \\
& + (1 - \lambda - \beta(x+1) - \mu)(V_n(1,x,y+2) + V_n(1,x+1,y) - V_n(1,x,y+1) - V_n(1,x+1,y+1)) \\
& + \beta(V_n(1,x,y+2) - V_n(1,x,y+1)) \geq \beta(V_n(1,x,y+2) + V_n(1,x,y) - 2V_n(1,x,y+1)) \geq 0,
\end{aligned}
$$

since Relations (7) and (8) are true for V_n and V_n is convex in y. Therefore, Relation (7) is true for D_n.

- For Relation (8), we have for $y \geq 0$,

$$
\begin{aligned}
& D_n(0,0,y+2) + D_n(1,0,y) - D_n(0,0,y+1) - D_n(1,0,y+1) \\
& = \mu(V_n(0,0,y) - V_n(0,0,y+1)) + (1 - \lambda - \mu)(V_n(0,0,y+2) + V_n(1,0,y) - V_n(0,0,y+1) - V_n(1,0,y+1)) \\
& + \mu(V_n(0,0,y+2) - V_n(0,0,y+1)) \geq \mu(V_n(0,0,y+2) + V_n(0,0,y) - 2V_n(0,0,y+1)) \geq 0,
\end{aligned}
$$

since Relation (8) is true for V_n and V_n is convex in y. Hence, Relation Relation (8) is true for D_n. The optimal policy is therefore a threshold policy.

We next prove that non-idling is optimal. We consider a threshold policy on the number of queued outbounds. We denote by y' the threshold on queue 2 such that the service of outbounds is allowed only if queue 2 has at least y' outbounds ($y' > 0$). The threshold y' is such that if $0 \leq y < y'$ no transition is possible from state (z, x, y) to any other state with strictly less than y outbounds for $z = 0, 1$, $x \geq 0$ and $0 \leq y < y'$. Therefore, all states with strictly less than y' outbound calls in queue 2 are transient states.

Hence, the probability to be in such a state is zero in the long run. In the stationary regime, the system always contains at least $y'-1$ outbounds in queue 2 since the server becomes idle after a service completion if $y=y'-1$. Therefore, all stationary performance measures related to inbounds are insensitive to y'. The only effect of y' is to force the system to contain an inventory of at least $y'-1$ waiting outbounds. Hence, the expected waiting time for outbounds increases with y'. It then follows that the optimal choice to solve the optimization problem ($\min SC$) is to choose $y'=1$. In other words, non-idling is optimal. The proof is completed. □

2 Proof of Corollary 2

For $0 \le x \le y_0$ we have

$$\lambda p_{x,0} = (x+1)\mu p_{x+1,0}.$$

Thus

$$p_{y_0,0} = \frac{a^{y_0}}{y_0!} p_{0,0}.$$

We denote by $P_x = \sum_{i=0}^{\infty} p_{x,i}$ for $x \ge y_0$. We may write for $y_0 \le x < s$,

$$\lambda P_x = (x+1)\mu P_{x+1}.$$

Then,

$$P_{y_0+x} = \frac{a^x y_0!}{(y_0+x)!} P_{y_0},$$

for $0 \le x \le s - y_0$. We also have

$$P_{s+x} = \frac{(a(1-\alpha))^x}{s^x} P_s,$$

for $0 \le x \le k$, and

$$P_{s+k+x} = \frac{((1-q-\alpha)a)^x}{s^x} P_{s+k},$$

for $x \ge 0$. Knowing that $y_0\mu(P_{y_0} - p_{y_0,0}) = \lambda q \sum_{x=0}^{\infty} P_{s+k+x}$, we obtain

$$P_{y_0} = \frac{\frac{a^{y_0}}{y_0!} p_{0,0}}{1 - q\frac{a}{y_0}\frac{a^{s-y_0}y_0!}{s!}\frac{(a(1-\alpha))^k}{s^k}\frac{1}{1-\frac{a(1-q-\alpha)}{s}}}.$$

Since all system state probabilities sum up to one, we deduce that

$$p_{0,0} = \left[\sum_{x=0}^{y_0-1}\frac{a^x}{x!} + \frac{\frac{a^{y_0}}{y_0!}\left(\sum_{x=0}^{s-y_0-1}\frac{a^x y_0!}{(y_0+x)!} + \frac{a^{s-y_0}y_0!}{s!}\sum_{x=0}^{k-1}\frac{(a(1-\alpha))^x}{s^x} + \frac{a^{s-y_0}y_0!}{s!}\frac{(a(1-\alpha))^k}{s^k}\frac{1}{1-\frac{a(1-q-\alpha)}{s}}\right)}{1 - q\frac{a}{y_0}\frac{a^{s-y_0}y_0!}{s!}\frac{(a(1-\alpha))^k}{s^k}\frac{1}{1-\frac{a(1-q-\alpha)}{s}}}\right]^{-1}.$$

Having in hand the system state stationary probabilities, we may write

$$\Psi = q\sum_{x=0}^{\infty} P_{s+k+x} = \frac{qP_{s+k}}{1-\frac{a(1-q-\alpha)}{s}}.$$

Using Little law, we also have $\lambda(1-\Psi-P_b)E(W_1)=\sum\limits_{x=0}^{\infty}xP_{s+x}$. Then,

$$\lambda(1-\Psi-P_b)E(W_1)=\sum_{x=0}^{k-1}xP_{s+x}+\sum_{x=0}^{\infty}(x+k)P_{s+k+x}=P_s\sum_{x=0}^{k-1}x\frac{(a(1-\alpha))^x}{s^x}+P_{s+k}\left(\frac{k\left(1-\frac{a(1-q-\alpha)}{s}\right)+\frac{a(1-q-\alpha)}{s}}{\left(1-\frac{a(1-q-\alpha)}{s}\right)^2}\right),$$

which finishes the proof of the corollary. □

3 Proof of Proposition 3

We define P_x as $P_x=\sum\limits_{y=0}^{\infty}p_{x,y}$ for $x\geq s$. Because of the non-idling policy, we do not allow states where $y>0$ and $x<s$.

We have $x\mu p_{x,0}=\lambda p_{x-1,0}$ for $1\leq x\leq s$, $P_{s+x}(s\mu+x\beta)=(1-\alpha_{x-1})P_{s+x-1}$ for $1\leq x\leq k$ and $P_{s+k+x}(s\mu+(x+k)\beta)=(1-\alpha_{k+x-1}-q_{k+x-1})P_{s+k+x-1}$ for $x\geq 1$. Therefore, $p_{s,0}=\frac{(\lambda/\mu)^s}{s!}p_{0,0}$, $P_{s+x}=P_s\lambda^x\prod\limits_{i=1}^{x}\left(\frac{1-\alpha_{i-1}}{s\mu+i\beta}\right)$ for $0\leq x\leq k$, and $P_{s+k+x}=P_s\lambda^{x+k}\frac{\prod\limits_{i=1}^{k}(1-\alpha_{i-1})\prod\limits_{i=k+1}^{x+k}(1-\alpha_{i-1}-q_{i-1})}{\prod\limits_{i=1}^{x+k}(s\mu+i\beta)}$ for $x\geq 0$.

Observing that $s\mu(P_s-p_{s,0})=\lambda\sum\limits_{x=0}^{\infty}q_{k+x}P_{s+k+x}$, we deduce that

$$P_s=\frac{\frac{a^s}{s!}}{1-\frac{a}{s}\prod\limits_{i=1}^{k}(1-\alpha_{i-1})\sum\limits_{x=0}^{\infty}\frac{q_{k+x}\lambda^{x+k}\prod\limits_{i=k+1}^{x+k}(1-\alpha_{i-1}-q_{i-1})}{\prod\limits_{i=1}^{x+k}(s\mu+i\beta)}}p_{0,0}.$$

Since all system state probabilities sum up to one, we may write

$$p_{0,0}=\left[\sum_{x=0}^{s-1}\frac{a^x}{x!}+\frac{\frac{a^s}{s!}\left(\sum\limits_{x=0}^{k}\lambda^x\prod\limits_{i=1}^{x}\left(\frac{1-\alpha_{i-1}}{s\mu+i\beta}\right)+\sum\limits_{x=1}^{\infty}\lambda^{x+k}\frac{\prod\limits_{i=1}^{k}(1-\alpha_{i-1})\prod\limits_{i=k+1}^{x+k}(1-\alpha_{i-1}-q_{i-1})}{\prod\limits_{i=1}^{x+k}(s\mu+i\beta)}\right)}{1-\frac{a}{s}\prod\limits_{i=1}^{k}(1-\alpha_{i-1})\sum\limits_{x=0}^{\infty}\frac{q_{k+x}\lambda^{x+k}\prod\limits_{i=k+1}^{x+k}(1-\alpha_{i-1}-q_{i-1})}{\prod\limits_{i=1}^{x+k}(s\mu+i\beta)}}\right]^{-1}.$$

We now use the stationary probabilities to derive the system performance measures. The proportion of customers who ask for the callback offer is

$$\Psi=\sum_{x=0}^{\infty}q_{k+x}P_{s+k+x}=\frac{s}{a}(P_s-p_{s,0})=\frac{\frac{a^s}{s!}\prod\limits_{i=1}^{k}(1-\alpha_{i-1})\sum\limits_{x=0}^{\infty}\frac{q_{k+x}\lambda^{x+k}\prod\limits_{i=k+1}^{x+k}(1-\alpha_{i-1}-q_{i-1})}{\prod\limits_{i=1}^{x+k}(s\mu+i\beta)}}{1-\frac{a}{s}\prod\limits_{i=1}^{k}(1-\alpha_{i-1})\sum\limits_{x=0}^{\infty}\frac{q_{k+x}\lambda^{x+k}\prod\limits_{i=k+1}^{x+k}(1-\alpha_{i-1}-q_{i-1})}{\prod\limits_{i=1}^{x+k}(s\mu+i\beta)}}p_{0,0}.$$

The proportion of customers who balk from queue 1 is

$$P_b=\sum_{x=0}^{\infty}\alpha_xP_{s+x}=\frac{\frac{a^s}{s!}\left(\sum\limits_{x=0}^{k}\alpha_x\lambda^x\prod\limits_{i=1}^{x}\left(\frac{1-\alpha_{i-1}}{s\mu+i\beta}\right)+\sum\limits_{x=1}^{\infty}\alpha_{k+x}\prod\limits_{i=1}^{k}(1-\alpha_{i-1})\frac{q_{k+x}\lambda^{x+k}\prod\limits_{i=k+1}^{x+k}(1-\alpha_{i-1}-q_{i-1})}{\prod\limits_{i=1}^{x+k}(s\mu+i\beta)}\right)}{1-\frac{a}{s}\prod\limits_{i=1}^{k}(1-\alpha_{i-1})\sum\limits_{x=0}^{\infty}\frac{q_{k+x}\lambda^{x+k}\prod\limits_{i=k+1}^{x+k}(1-\alpha_{i-1}-q_{i-1})}{\prod\limits_{i=1}^{x+k}(s\mu+i\beta)}}p_{0,0}.$$

The expected number of customers in queue 1, say $E(N_1)$, is

$$E(N_1) = \sum_{x=0}^{\infty} x P_{s+x} = \frac{\frac{a^s}{s!}\left(\sum_{x=0}^{k} x\lambda^x \prod_{i=1}^{x}\left(\frac{1-\alpha_{i-1}}{s\mu+i\beta}\right) + \sum_{x=1}^{\infty}(k+x)\prod_{i=1}^{k}(1-\alpha_{i-1})\frac{q_{k+x}\lambda^{x+k}\prod_{i=k+1}^{x+k}(1-\alpha_{i-1}-q_{i-1})}{\prod_{i=1}^{x+k}(s\mu+i\beta)}\right)}{1-\frac{a}{s}\prod_{i=1}^{k}(1-\alpha_{i-1})\sum_{x=0}^{\infty}\frac{q_{k+x}\lambda^{x+k}\prod_{i=k+1}^{x+k}(1-\alpha_{i-1}-q_{i-1})}{\prod_{i=1}^{x+k}(s\mu+i\beta)}} p_{0,0}.$$

We next deduce that

$$P_a = \frac{\frac{a^s}{s!}\left(\sum_{x=0}^{k}\left(\alpha_x + x\frac{\beta}{\lambda}\right)\lambda^x \prod_{i=1}^{x}\left(\frac{1-\alpha_{i-1}}{s\mu+i\beta}\right) + \sum_{x=1}^{\infty}\left(\alpha_{x+k}+(x+k)\frac{\beta}{\lambda}\right)\prod_{i=1}^{k}(1-\alpha_{i-1})\frac{q_{k+x}\lambda^{x+k}\prod_{i=k+1}^{x+k}(1-\alpha_{i-1}-q_{i-1})}{\prod_{i=1}^{x+k}(s\mu+i\beta)}\right)}{1-\frac{a}{s}\prod_{i=1}^{k}(1-\alpha_{i-1})\sum_{x=0}^{\infty}\frac{q_{k+x}\lambda^{x+k}\prod_{i=k+1}^{x+k}(1-\alpha_{i-1}-q_{i-1})}{\prod_{i=1}^{x+k}(s\mu+i\beta)}} p_{0,0},$$

and using Little law, the expected waiting time in queue 1 (for served customers and customers who abandon after experiencing some wait) is given by

$$\begin{aligned} E(W_1) &= \frac{E(N_1)}{\lambda(1-\Psi-P_b)} \\ &= \frac{1}{\lambda}\frac{\frac{a^s}{s!}\left(\sum_{x=0}^{k} x\lambda^x \prod_{i=1}^{x}\left(\frac{1-\alpha_{i-1}}{s\mu+i\beta}\right) + \sum_{x=1}^{\infty}(x+k)\prod_{i=1}^{k}(1-\alpha_{i-1})\frac{q_{k+x}\lambda^{x+k}\prod_{i=k+1}^{x+k}(1-\alpha_{i-1}-q_{i-1})}{\prod_{i=1}^{x+k}(s\mu+i\beta)}\right)}{1-\frac{a^s}{s!}\sum_{x=0}^{k-1}\alpha_x\lambda^x\prod_{i=1}^{x}\left(\frac{1-\alpha_{i-1}}{s\mu+i\beta}\right)p_{0,0} + \sum_{x=0}^{\infty}\left(\alpha_{k+x}\frac{a^s}{s!}p_{0,0}+\frac{a}{s}(p_{0,0}-1)\right)\prod_{i=1}^{k}(1-\alpha_{i-1})\frac{q_{k+x}\lambda^{x+k}\prod_{i=k+1}^{x+k}(1-\alpha_{i-1}-q_{i-1})}{\prod_{i=1}^{x+k}(s\mu+i\beta)}} p_{0,0}. \end{aligned}$$

This completes the proof of the proposition. □

References

Koole, G. (2007). *Monotonicity in Markov reward and decision chains: Theory and applications*, volume 1. Now Publishers Inc.